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Hamilton decompositions of 6-regular abelian Cayley graphs

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HAMILTON DECOMPOSITIONS OF 6-REGULAR ABELIAN CAYLEY GRAPHS

By

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A DISSERTATION

Submitted in partial fulfillment of the requirements

for the degree of

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This dissertation, "Hamilton Decompositions of 6-Regular Abelian Cayley Graphs", is hereby approved in partial fulfillment of the requirements for the degree of DOCTOR OF PHILOSOPHY in the field of Mathematical Sciences.

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To my family.

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Abstract

In 1969, Lovász asked whether every connected, vertex-transitive graph has a Hamilton path. This question has generated a considerable amount of interest, yet remains vastly open. To date, there exist no known connected, vertex-transitive graph that does not possess a Hamilton path. For the Cayley graphs, a subclass of vertex-transitive graphs, the following conjecture was made:

Weak Lovász Conjecture: Every nontrivial, finite, connected Cayley graph is hamiltonian.

The Chen-Quimpo Theorem proves that Cayley graphs on abelian groups flourish with Hamilton cycles, thus prompting Alspach to make the following conjecture:

Alspach Conjecture: Every $2k$-regular, connected Cayley graph on a finite abelian group has a Hamilton decomposition.

Alspach's conjecture is true for $k = 1$ and 2, but even the case $k = 3$ is still open. It is this case that this thesis addresses.

Chapters 1–3 give introductory material and past work on the conjecture. Chapter 3 investigates the relationship between 6-regular Cayley graphs and associated quotient graphs. A proof of Alspach's conjecture is given for the odd order case when $k = 3$. Chapter 4 provides a proof of the conjecture for even order graphs with 3-element connection sets that have an element generating a subgroup of index 2, and having a linear dependency among the other generators.

Chapter 5 shows that if $\Gamma = \textsc{Cay}(A, \{s_1, s_2, s_3\})$ is a connected, 6-regular, abelian Cayley graph of even order, and for some $1 \leqslant i \leqslant 3$, $\Delta_i = \textsc{Cay}(A/\langle s_i\rangle, \{\overline{s_{j_1}}, \overline{s_{j_2}}\})$ is 4-regular, and $\Delta_i \not\cong \textsc{Cay}(\mathbb{Z}_3, \{1, 1\})$, then Γ has a Hamilton decomposition. Alternatively stated, if $\Gamma = \textsc{Cay}(A, S)$ is a connected, 6-regular, abelian Cayley graph of even order, then Γ has a Hamilton decomposition if S has no involutions, and for some $s \in S$, $\textsc{Cay}(A/\langle s\rangle, \overline{S})$ is 4-regular, and of order at least 4.

Finally, the Appendices give computational data resulting from C and MAGMA programs used to generate Hamilton decompositions of certain non-isomorphic Cayley graphs on low order abelian groups.

Chapter 1

Cayley Graphs and Hamilton Cycles

1.1 Overview

This thesis discusses the classical problem in graph theory of finding Hamilton cycles, a cycle that visits each vertex once, in finite simple graphs. The search for necessary and sufficient conditions for the existence of Hamilton paths, cycles, and decompositions in graphs, digraphs, and hypergraphs, is as old as graph theory itself, and many questions yet remain. As far as the author is aware, there is considerable dissension among graph theorists as to whether there even exist good characterizations for hamiltonicity in general, and to complicate things, this problem falls in the set of NP-complete problems of computational complexity theory. We consider the problem of obtaining Hamilton decompositions, partitions of the edge set of a graph into Hamilton cycles, by restricting attention to graphs that are based on an algebraic group, i.e., the family of Cayley graphs.

We begin with an introduction to terminology, definitions, and notation, which can be found in standard textbooks on graph theory, such as [52] and [17]. We then proceed to develop a framework of theorems that answer, in the affirmative, a subcase of a conjecture of Alspach, stemming from questions of Lovász, Parsons, and many others, on Hamilton cycles and decompositions of Cayley graphs on finite abelian groups.

1.2 Preliminaries

Graphs are a vast class of combinatorial structures and are ubiquitous in that they are used to describe relationships. Graphs are used to model ecosystems, phylogenetic trees, and protein-protein interactions in biology; network flows, routing problems, and data structures in computer science and engineering; molecular structure in organic chemistry; countless problems from combinatorics, abstract algebra, matrix algebra, recreational mathematics, probability theory, and statistics.

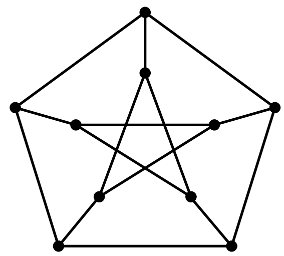


Figure 1.1: The famous Petersen graph is a 3-regular vertex-transitive graph.

A *graph* Γ is a pair of sets (V, E), where $V = V(\Gamma)$ is called the *vertex set* (or node set or point set) and $E = E(\Gamma) = \{\{x, y\} : x, y \in V(\Gamma)\}$ is called the *edge set*. Elements of E are called *edges* of Γ. For example, a vertex set could be the set of all airports located in U.S. cities, where vertices x and y have an edge between them, denoted $\{x, y\}$, if one city has a direct flight to the other. The cardinality of V is called the *order* of the graph, and the cardinality of E is called the *size* of the graph.

If $\{x, y\} \in E$, it is common to write xy in place of $\{x, y\}$ when the context is clear, and we say x is *adjacent* to y. If $e = \{x, y\}$, we say x and y are *incident* to e. If E is a multiset, we say Γ is a *multigraph*. Any edge of the form $\{x, x\}$ for some $x \in V(\Gamma)$, is called a *loop*. A *simple* graph is a graph without multiple edges or loops. Most of the graphs in this thesis will be simple. Two graphs Γ_1 and Γ_2 are *equal* if and only if $V(\Gamma_1) = V(\Gamma_2)$ and $E(\Gamma_1) = E(\Gamma_2)$. Given a graph $\Gamma = (V, E)$, the *complement* of Γ, denoted $\overline{\Gamma}$ is the graph with vertex $V(\Gamma)$, and edge xy if and only if $xy \notin E(\Gamma)$.

More commonly, we are interested in when two graphs *behave essentially the same*. The graphs Γ_1 and Γ_2 are said to be *homomorphic*,, if there exists a mapping $\varphi : V(\Gamma_1) \to V(\Gamma_2)$, such that $\{x, y\} \in E(\Gamma_1)$ if and only if $\{\varphi(x), \varphi(y)\} \in E(\Gamma_2)$. I.e., any homomorphism maps edges to edges and nonedges to nonedges. A graph homomorphism φ that is a bijective map is called an *isomorphism*, denoted $\Gamma_1 \simeq \Gamma_2$. Order, size, degree sequence, cycle structure, and many other parameters are invariant under isomorphism. An isomorphism from a graph to itself is called an *automorphism* of Γ. The set of all automorphisms of Γ form an algebraic group, called it *automorphism group*, denoted $\textsc{Aut}(\Gamma)$. For example, $\textsc{Aut}(K_n) \cong S_n$, the group of all permutations of n elements. It is usually a very difficult problem to determine the automorphism group of a graph, though it pays large dividends in terms of understanding the structure of the graph. It is readily seen that $\textsc{Aut}(\Gamma)$ permutes the set of vertices of common degree r among themselves.

A graph Γ is called *vertex-transitive* if for any $x, y \in V(\Gamma)$, there exists $\varphi \in \textsc{Aut}(\Gamma)$ such that $\varphi(x) = y$. Vertex-transitive graphs are necessarily regular. For example, the well-known Petersen graph shown in Figure 1.1 is a vertex-trasitive graph with automorphism group S_5. An important class of vertex-transitive graphs, called Cayley graphs, are introduced in Section 1.4. Cayley graphs, named after the mathematician Arthur Cayley, encode group-theoretic structure, are used extensively in combinatorial group theory.

The *union* of the graphs Γ_1 and Γ_2, denoted $\Gamma_1 \cup \Gamma_2$ is the graph with vertex set $V(\Gamma_1 \cup \Gamma_2) = V(\Gamma_1) \cup V(\Gamma_2)$ and edge set $E(\Gamma_1 \cup \Gamma_2) = E(\Gamma_1) \cup E(\Gamma_2)$. Similarly, the *intersection* of the graphs Γ_1 and Γ_2, denoted $\Gamma_1 \cap \Gamma_2$ is the graph with vertex set $V(\Gamma_1 \cap \Gamma_2) = V(\Gamma_1) \cap V(\Gamma_2)$ and edge set

$E(\Gamma_1 \cap \Gamma_2) = E(\Gamma_1) \cap E(\Gamma_2)$. If $E(\Gamma_1 \cap \Gamma_2) = \emptyset$, then Γ_1 and Γ_2 are said to be *edge-disjoint*. If $V(\Gamma_1 \cap \Gamma_2) = \emptyset$, then Γ_1 and Γ_2 are said to be *vertex-disjoint* (any two vertex-disjoint graphs are also edge-disjoint). If $\Gamma_1 \cap \Gamma_2 = \emptyset$, the empty graph, then Γ_1 and Γ_2 are said to be *disjoint*.

If the vertex x of Γ is adjacent to r other vertices, we say x *has degree* r, denoted $\deg_\Gamma(x) = r$. If every vertex of Γ has common degree r, then Γ is said to be r*-regular* . Furthermore, Γ is simply said to be *regular* if it is r-regular for some $r \geqslant 0$. A finite simple graph Γ of order n in which every pair of vertices are joined by an edge, i.e., $|V(\Gamma)| = n$ and $|E(\Gamma)| = \binom{n}{2}$, is called a *complete* graph. This is denoted $\Gamma = K_n$, and note that complete graphs are $(n-1)$-regular. 3-regular graphs are called *cubic* graphs and 4-regular graphs are sometimes called *quartic* graphs. The *minimum degree* of Γ is $\delta(\Gamma) = \min\{\deg(x) : x \in \Gamma\}$ and the *maximum degree* of Γ is $\Delta(\Gamma) = \max\{\deg(x) : x \in \Gamma\}$.

Any alternating sequence W of (not necessarily distinct) vertices and edges in Γ,

$$W := x_0 e_0 x_1 e_1 \cdots x_{k-1} e_{k-1} x_k, \quad e_i = \{x_i, x_{i+1}\} \quad 0 \leqslant i \leqslant k-1,$$

is called a *walk (of length k)*. x_0 is called the *initial vertex* and x_k, the *terminal vertex*. Any walk with distinct vertices is called a *path*. If $x_0 = x_k$, the walk is said to be *closed*. A closed path of length $k \geqslant 3$ is called a k*-cycle*, usually denoted C_k. A graph is *connected* if there exists a path between any two vertices. Determining how connected a graph is is a central topic in graph theory. We say Γ is k*-connected*, provided that we may delete any $k-1$ or fewer vertices from Γ and it will remain connected. The greatest integer k, such that Γ is k-connected, is called the *vertex-connectivity*, or just *connectivity*, of Γ, and is denoted $\kappa(\Gamma)$. Any cycle is a connected 2-regular graph. The smallest cycle a graph can have is K_3, the triangle. The length of the smallest cycle in a graph Γ is called its *girth*, denoted $g(\Gamma)$.

A *subgraph* Δ of a graph Γ is a graph where $V(\Delta) \subseteq V(\Gamma)$ and $E(\Delta) \subseteq E(\Gamma)$. If $V(\Delta) = V(\Gamma)$, Δ is a *spanning subgraph* of Γ. A k*-factor* of Γ is a k-regular spanning subgraph of Γ. If $V_0 \subseteq V(\Gamma)$, then the subgraph of Γ *induced* by V_0, is defined to be the graph $\Gamma[V_0]$ with vertex set V_0, and all edges $xy \in E(\Gamma)$ with $x, y \in V_0$. The *independence number* of a graph Γ, denoted $\alpha(\Gamma)$, is the largest number of vertices, such that their induced subgraph forms a *coclique*, i.e., a graph with no edges.

If $E_0 \subseteq E(\Gamma)$, and $E_0 \neq \emptyset$, then the subgraph of Γ *edge-induced* by E_0, is denoted $\langle E_0 \rangle$, and is defined to be the graph having as vertex set all vertices of Γ which are incident with at least one edge in E_0, and whose edge set is E_0. Given a graph Γ, a partition of $E(\Gamma)$ into subsets so that

$$E(\Gamma) = E_1 \,\cup\, E_2 \,\cup\, \cdots \,\cup\, E_t$$

where $|E_i| = |E_j|$ and $E_i \cap E_j = \emptyset$ for all $i \neq j$ is called an *isomorphic factorization* of Γ provided the t subgraphs edge-induced on the sets $\langle E_i \rangle$, $i = 1, 2, \ldots, t$ are pairwise isomorphic. If Γ has an isomorphic factorization into subgraphs with k edges (each being isomorphic to the graph H), we say Γ has a k*-isofactorization into the graph* H. See Alspach, Dyer, Kreher [5] and Kreher-Westlund [26], for recent results in isofactorizations of circulant graphs.

Let $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ be graphs. The *cartesian product* (or *box product*) of Γ_1 and Γ_2, denoted, $\Gamma = \Gamma_1 \,\Box\, \Gamma_2$ is the graph with vertex set $V(\Gamma) = V_1 \times V_2 = \{(v_1, v_2) : v_1 \in V_1 \text{ and } v_2 \in V_2\}$, and edge set

$$E(\Gamma) = \{\{(u_1, u_2), (u_1, v_2)\} : \ \{u_2, v_2\} \in E_2\} \cup \{\{(u_1, u_2), (v_1, u_2)\} : \ \{u_1, v_1\} \in E_1\}.$$

1.3 Hamilton Cycles

If Γ has order n, then any path of length $n-1$ (a path using every vertex once) is called a *Hamilton path*. A graph is called *hamilton-connected* (or *strongly hamiltonian*) if every two vertices are joined by a Hamilton path. Any closed walk using every edge once is called a *Hamilton cycle*. Any graph that contains at least one Hamilton cycle is said to be *hamiltonian*. Clearly, a graph must be connected to possibly be hamiltonian and any Hamilton cycle is a connected 2-factor. A bipartite graph with bipartition A and B, where $|A| = |B|$, is said to be *hamilton-laceable* if, for all $a \in A$, $b \in B$, there exists a Hamilton path from a to b.

Hamilton cycles are named after Sir William Rowan Hamilton, an Irish mathematician who invented a puzzle called the *Icosian game*, that involves finding Hamilton cycles on the edge graph of the dodecahedron (see Figure 1.2). In general, there is no known good characterization for graphs that are hamiltonian, in part due to the fact that the decision problem of determining the existence of a Hamilton path (or cycle) in a graph is one of the classical NP-complete problems of computational complexity theory. A simple sufficiency condition for the existence of Hamilton cycles was proved by Dirac in 1952.

Theorem 1.3.1 (Dirac [18])**.** *Every graph Γ of order $n \geqslant 3$ where $\delta(\Gamma) \geqslant n/2$ is hamiltonian.*

Dirac's theorem was generalized by Ore in 1960.

Theorem 1.3.2 (Ore [46])**.** *Every graph, Γ, of order $n \geqslant 3$ is hamiltonian if for all $x, y \in V(\Gamma)$,*

$$\deg(x) + \deg(y) \geqslant n.$$

Given a graph Γ of order n, the *closure* of Γ is the graph with the same vertex set as Γ that is obtained from Γ by iteratively adding an edge $\{x, y\}$ for all nonadjacent pairs of vertices x and y satisfying $\deg(x) + \deg(y) \geqslant n$, until no such pair remains. The closure is well-defined, as the order in which edges are added does not matter.

Theorem 1.3.3 (Bondy-Chvátal [10])**.** *A simple graph of order n is hamiltonian if and only if its closure is also hamiltonian.*

Another, well-known result, proved in 1972 by Chvátal and Erdös, relates the independence number and vertex-connectivity to determine when a graph is hamiltonian.

Theorem 1.3.4 (Chvátal-Erdös [13])**.** *Every graph, Γ, of order at least three, is hamiltonian if*

$$\alpha(\Gamma) \leqslant \kappa(\Gamma).$$

Given a graph $\Gamma = (V, E)$, define the graph $\Gamma^d = (V', E')$ to be the graph where $V = V'$ and $\{x, y\} \in E'$ if and only if the shortest path between x and y has length d.

Theorem 1.3.5 (Fleischner [22, 23])**.** *If Γ is a 2-connected graph, then G^2 is hamiltonian.*

An excellent survey of recent developments on the Hamilton Problem is Gould [24].

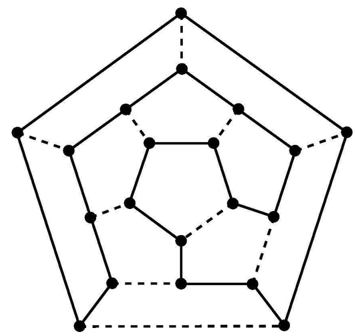


Figure 1.2: One solution to Hamilton's puzzle.

1.3.1 Hamilton decompositions

If a graph Γ is $2k$-regular, any partition of $E(\Gamma)$ into k Hamilton cycles (if one exists) is called a *Hamilton decomposition*. If Γ is $(2k+1)$-regular, a Hamilton decomposition is defined to be any partition of $E(\Gamma)$ into k Hamilton cycles and a 1-factor (a 1-regular spanning subgraph of Γ or perfect matching). A well-studied general conjecture on sufficient conditions for a Hamilton decomposition is the Nash-Williams Conjecture.

Nash-Williams Conjecture [45]. Every k-regular graph with a most $2k+1$ vertices has a Hamilton decomposition.

In 1974, Kotzig proved that every cartesian product of two cycles, $C_a \,\square\, C_b$, has a Hamilton decomposition. The following 1982 theorem generalized this result.

Theorem 1.3.6 (Aubert-Schneider [7]). *If Γ can be decomposed into two Hamilton cycles and C is a cycle, then $\Gamma \,\square\, C$ can be decomposed into three Hamilton cycles.*

In 1990, Alspach, Bermond, and Sotteau generalized Kotzig's result to the cartesian product of any finite number of cycles.

Theorem 1.3.7 (Alspach, Bermond, Sotteau [4]). *$C_{\ell_1} \,\square\, C_{\ell_2} \,\square\, \cdots \,\square\, C_{\ell_t}$ has a Hamilton decomposition, for all integers ℓ_i, where $1 \leqslant i \leqslant t$.*

In 1991, Stong provided the following generalization.

Theorem 1.3.8 (Stong [50]). *If Γ_1 and Γ_2 are decomposable into n and m Hamilton cycles, respectively, with $n \geqslant m$, then $\Gamma_1 \,\square\, \Gamma_2$ has a Hamilton decomposition if one of the following holds:*

1. *$m \leqslant 3n$,*
2. *$n \geqslant 3$,*
3. *$|V(\Gamma_1)|$ is even, or*
4. *$|V(\Gamma_2)| \geqslant 6\lceil m/n \rceil - 3$*

1.4 Cayley Graphs

Much of the material in this section can be found in [8]. A Cayley graph, $\Gamma = (V, E)$, is a graph whose vertex set is a group G. Although it is perfectly reasonable to take G to be of infinite order, we are primarily interested in a finite number of vertices. Let $S \subseteq G \setminus \{1\}$ be a subset of non-identity elements of G, such that $S = S^{-1}$. S is called the *connection set* of Γ, and for brevity, we usually write $S = \{s_1, s_2, \ldots, s_k\}$ in place of $S = \{s_1, s_1^{-1}, s_2, s_2^{-1}, \ldots, s_k, s_k^{-1}\}$. The edge set of Γ is defined to be

$$E(\Gamma) = \{\{x, y\} : yx^{-1} \in S\}.$$

The requirement that $1 \notin S$ ensures there are no loops in Γ, and the requirement that $S^{-1} = S$ ensures that Γ is undirected. A *Cayley digraph*, denoted $\overrightarrow{\mathrm{Cay}}(G, S)$, is a graph with vertex set G, and arc set $E = \{(x, y) : yx^{-1} \in S\}$. If $yx^{-1} = s \in S$, the edge $\{x, y\}$ is said to be *generated by* s. A subgraph H is generated by s if every edge in H is generated by s. Γ is a connected graph if and only if S generates G. Let $|s|$ denote the additive order of the element s. Any involution s, an element of order 2, generates a 1-factor (or 1-regular spanning subgraph) and if $|s| > 2$, s generates a 2-factor of G. If $S = \{s_1, \ldots, s_k\}$ and $|s_i| > 2$ for all $1 \leqslant i \leqslant k$, then Γ is $2k$-regular. If G is a cyclic group, i.e. $G \cong \mathbb{Z}_n$, then Γ is called a *circulant graph*. The adjacency matrix of a circulant graph is a circulant matrix. Regarding the isomorphism problem for circulant graphs, the following is known:

Theorem 1.4.1 (Turner [51]). *For any prime p, $\mathrm{Cay}(\mathbb{Z}_p, S) \cong \mathrm{Cay}(\mathbb{Z}_p, S') \Leftrightarrow S' = aS$, for some $a \in \mathbb{Z}_p^*$.*

The family of k-bit binary cubes, or k-cubes used in the construction of reflected Grey codes are Cayley graphs. The underlying group is the elementary abelian 2-group $\mathbb{Z}_2^k$ and S is the standard generating set, $S = \{e_1, e_2, \ldots, e_k\}$ where $e_i := (e'_1, \ldots, e'_k)$, $e'_i = 1$, and $e'_j = 0$ for all $j \neq i$. Figure 1.3 illustrates the circulant graph $\mathrm{Cay}(\mathbb{Z}_{18}, \{2, 6, 9\})$ and a 4-bit binary cube $\mathrm{Cay}(\mathbb{Z}_2^4, \{(1, 0, 0, 0,), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\})$.

The following well-known result places Cayley graphs inside the family of vertex-transitive graphs.

Theorem 1.4.2. *Every Cayley graph is vertex-transitive.*

Proof. For each $g \in G$, the map $\varphi_g : G \to G$ defined by $\varphi_g : x \mapsto xg$ is bijection on G. Also $\varphi_g \in \mathrm{Aut}(\mathrm{Cay}(G, S))$ as

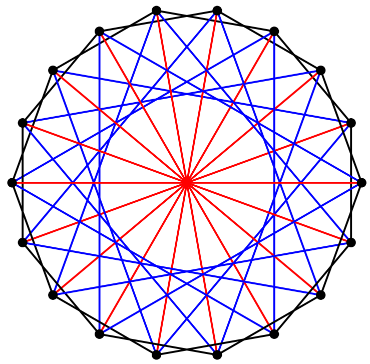
$$\{x, y\} \in E(\Gamma) \Leftrightarrow yx^{-1} = s \Leftrightarrow ygg^{-1}x = (yg)(xg)^{-1} = s \Leftrightarrow \{\varphi_g(x), \varphi_g(y)\} \in E(\Gamma)$$

The group $G_R = \{\varphi_g : g \in G\}$ is a transitive subgroup of $\mathrm{Aut}(\Gamma)$ as $\varphi_{x^{-1}y}(x) = y \; \forall \, x, y \in G$. ■

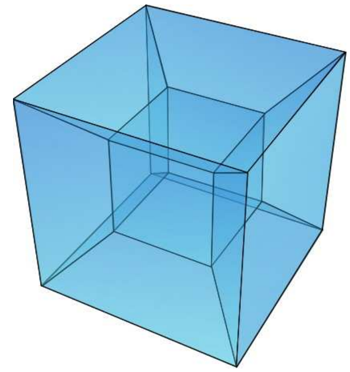
The Petersen graph, shown in Figure 1.1, is vertex-transitive graph that is not the Cayley graph of any group. This is a consequence of the following result that characterizes all Cayley graphs.

Theorem 1.4.3 (Sabidussi [48]). *A graph Γ is a Cayley graph if and only if $\mathrm{Aut}(\Gamma)$ contains a regular subgroup, i.e., a sharply 1-transitive subgroup.*

Proof. (Sketch). If $\Gamma = \mathrm{Cay}(G, S)$, then $R = \{\varphi_g : g \in G\} \leqslant \mathrm{Aut}(\Gamma)$ is regular, for $|G| = |R| \Rightarrow |R_g| = 1$. Conversely, if G is a regular subgroup of $\mathrm{Aut}(\Gamma)$, then label an arbitrary $x \in V(\Gamma)$ with $1 = 1_G$ and label $y \in V(\Gamma)$ with $g \in G$ such that $x^g = y$. Clearly, Γ is a Cayley graph on G. ■



$\mathrm{CAY}(\mathbb{Z}_{18}, \{2, 6, 9\})$



$\mathrm{CAY}(\mathbb{Z}_2^4, \{e_1, e_2, e_3, e_4\})$

Figure 1.3: A circulant graph and the 4-bit binary cube (hypercube)*.

The Petersen graph has automorphism group $\mathrm{AUT}(P) \cong S_5$, which possesses no transitive subgroup of order 10. See McKay-Praeger [44] for other vertex-transitive graphs that are not Cayley graphs. If φ is any automorphism of the group G, then $\mathrm{CAY}(G, S)$ and $\mathrm{CAY}(G, \varphi(S))$ are isomorphic. Additionally, the following is known.

Theorem 1.4.4 (Sabidussi [49])**.** *Every vertex-transitive graph is the homomorphic image of a Cayley graph.*

Theorem 1.4.5 (Folklore, Marušič [40])**.** *For any prime p, all vertex transitive graphs of order p, p^2, and p^3, or $2p$ (where $p \equiv 3 \pmod 4$) are Cayley graphs.*

1.4.1 Lovász Conjecture

Almost forty years ago, Lovász [36], Parsons, and others, posed the following research problem (though originally phrased in the negative): Which connected, vertex-transitive graphs have a Hamilton path?

Lovász Conjecture. *Every finite, connected, vertex-transitive graph has a Hamilton path.*

Theorem 1.4.6 ([1, 12, 27, 29, 40, 41, 42, 43, 39, 51])**.** *Every connected, vertex-transitive graph of order kp, p^k, or $2p^2$, where $k \leqslant 4$, and p a prime, is either hamiltonian, the Petersen graph, or the Coxeter graph. Furthermore, every connected, vertex-transitive graph of order $5p$ or $6p$ has a Hamilton path.*

This conjecture has generated a huge body of literature, and to date, there exists no known connected, vertex-transitive graph that does not possess a Hamilton path. Furthermore, of these, only four nontrivial vertex-transitive graphs exist that do not have a Hamilton cycle. They are the Petersen graph, the Coxeter graph, and the truncated Petersen and Coxeter graphs, all of which are hypohamiltonian. However, none of these are Cayley graphs. Thus, we have the following conjecture:

*The circulant graph of Figure 1.3 was drawn using *Mathematica*. The hypercube image is available for free use from *Wikipedia Commons* under the *GNU Free Documentation License*.

Weak Lovász Conjecture [36]. *Every non-trivial, connected, Cayley graph is hamiltonian.*

There are no known Cayley graphs which are non-hamiltonian. The Lovász conjecture is true for Cayley graphs on finite abelian groups, proved in [38], though it appeared earlier as a problem in [37]. We offer a proof of this important result to aid the reader.

Theorem 1.4.7 (Lovász [37]). *If Γ is a connected graph, and $\textsc{Aut}(\Gamma)$ contains an abelian, transitive subgroup, then Γ is hamiltonian.*

Proof. Let $H \leqslant \textsc{Aut}(\Gamma)$ be abelian, and act transitively on $V(\Gamma)$. If $h(x) = x$, for some $x \in V(\Gamma)$, and $h \in H$, then $\forall\, y \in V(\Gamma)$, there exists $g \in H$ such that $g(x) = y$, and so

$$h(y) = h(g(x)) = g(h(x)) = g(x) = y \Rightarrow h = 1.$$

Hence, $|V(\Gamma)| = |H(x)| = [H : H_x] = |H|$, and the action is regular. Consider all subgroups $H' \leqslant H$ with the property that $H'(x)$ forms a cycle. In particular, at least one such H' exists: take $h' \in H$ with $\{x, h'(x)\} \in E$. Then $H = \langle h' \rangle$ is a cycle. Let $K \leqslant H$ be maximal with respect to this property,

$$K(x) := x, k_1(x), k_2(x), \ldots, k_{d-1}(x), x.$$

We claim $d = |V(\Gamma)|$. If $d < |V(\Gamma)|$ then there exists $y \in V(\Gamma)$ such that $h(x) = y$, for some $h \in H \setminus K$, where $\{k^i(x), y\} \in E(\Gamma)$. Hence, $\{x, k^{-i}(y)\} \in E$. Let $\ell \in H$ satisfy $\ell(x) = k^{-i}(y)$. Clearly, for all $z \in V(\Gamma)$, $\{z, \ell(z)\} \in E$. Let α be the minimum integer such that $\ell^\alpha \in K$. As H is abelian, L is a subgroup of H, where

$$L = K \cup \ell K \cup \ell^2 K \cup \cdots \cup \ell^{\alpha-1} K.$$

Now, $P := x, \ell(x), \ell^2(x), \ldots, \ell^{\alpha-1}(x)$ is an α-path, and $P \,\Box\, K(x) \simeq P \,\Box\, C_d$ is subgraph of Γ, spanning $L(x)$, and is easily seen to be hamiltonian. This is a contradiction to K being maximal, thus Γ is hamiltonian. ∎

The result for abelian groups is also a corollary of the well-known Chen-Quimpo Theorem, for every edge lies on a Hamilton cycle.

Theorem 1.4.8 (Chen-Quimpo [11]). *Every connected Cayley graph, of degree at least three, on a finite abelian group, is hamilton-connected if not bipartite, or hamilton-laceable if bipartite.*

Theorem 1.4.9 (Durnberger [20], Marušič [38], Keating-Witte [25]). *Every Cayley graph on a group G is hamiltonian if it is prime-power order, or the commutator subgroup $[G, G] = \langle g^{-1}h^{-1}gh : g, h \in G\rangle$ is cyclic of prime-power order.*

Theorem 1.4.10 (Dobson et al. [19]). *Every connected graph of order at least 3, with a transitive group of automorphisms, whose commutator subgroup is cyclic of prime-power order, has a Hamilton cycle, or is the Petersen graph.*

However, there do exist infinite families of Cayley digraphs that do not have a directed Hamilton cycle. For example, there even exist infinite families of circulant digraphs that are non-hamiltonian.

Theorem 1.4.11 (Locke-Witte [35]). *There exists no Hamilton cycle in $\overrightarrow{\textsc{Cay}}(\mathbb{Z}_{12k}, \{6k, 6k+2, 6k+3\})$ or in $\overrightarrow{\textsc{Cay}}(\mathbb{Z}_{2k}, \{a, a+1, a+k\})$ if $a+k$ is even, and $\gcd(2k, a), \gcd(2k, a+k) > 1$.*

The following result of Witte answers the question for Cayley digraphs of prime power order:

Theorem 1.4.12 (Witte [54])**.** *Every connected Cayley digraph on a group of order p^α, where p is a prime, and $\alpha \geqslant 1$, has a directed Hamilton cycle.*

For more recent developments on the Lovász conjecture, see Kutnar-Marušič [28] and Pak-Radoičić [47]. For surveys on hamiltonicity properties of Cayley graphs, see Curran-Gallian [14] and Witte-Gallian [55].

1.5 Alspach Conjecture

One implication of the Chen-Quimpo Theorem is that Cayley graphs on abelian groups are not only hamiltonian, but *flourish* with Hamilton cycles. Thus, Alspach posed the following question:

Does every connected Cayley graph on an abelian group admit a Hamilton decomposition?

In [2], Alspach made the following conjecture.

Alspach Conjecture [2]. *Every $2k$-regular connected Cayley graph on a finite abelian group has a decomposition into k edge-disjoint Hamilton cycles.*

This is trivially true if $k = 1$, for any such graph is already a Hamilton cycle. Bermond et al. proved the conjecture true when $k = 2$.

Theorem 1.5.1 (Bermond et al. [9])**.** *Every 4-regular connected Cayley graph on a finite abelian group is decomposable into two Hamilton cycles.*

There are no known counterexamples to Alspach's conjecture, but even the case $k = 3$ is still open. In 2006, Dean proved the following for circulant graphs.

Theorem 1.5.2 (Dean [15, 16])**.** *If $\Gamma = \mathrm{Cay}(A, S)$ is a connected 6-regular circulant graph then Γ has a Hamilton decomposition if one of the following holds:*

1. *$|A|$ is odd, or*
2. *there exists $s \in S$, such that $\langle s \rangle = A$.*

Let $S := \{s_1, s_2, \ldots, s_k\}$ generate a finite group A. S is said to be *minimal generating set* if for each $1 \leqslant i \leqslant k$, the element $s_i \notin \langle s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_k\rangle$. Likewise, the set S is said to be *strongly minimal* if for all $1 \leqslant i \leqslant k$, the element $2s_i \notin \langle s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_k\rangle$. Clearly, every strongly minimal generating set is also a minimal generating set. Liu has used induction on the order of a quotient group to prove the following general result.

Theorem 1.5.3 (Liu [33, 34])**.** *Let $S = \{s_1, \ldots, s_k\}$ generate a finite abelian group A. Then $\mathrm{Cay}(A, S)$ has a Hamilton decomposition if one of the following holds:*

1. *the order of A is odd, and S is a minimal generating set, or*
2. *the order of A is even, and S is a strongly minimal generating set.*

Corollary 1.5.4 (Liu [34])**.** *Let S be a minimal connection set of A having even order at least four. If $\langle s \rangle$ has odd index, for all $s \in S$, then $\mathrm{Cay}(A, S)$ has a Hamilton decomposition.*

Theorem 1.5.5 (Li et al. [30])**.** *Any connected Cayley graph on an abelian group of order p^2 or pq, where p and q are odd primes, has a Hamilton decomposition.*

1.6 New Results

In Chapter 3, the following result is shown.

Theorem 3.2.1 (Kreher, Liu, Westlund [53]). *Every connected, 6-regular, abelian Cayley graph of odd order has a Hamilton decomposition.*

In Chapter 4, the following results are shown.

Theorem 4.1.5. *If $A = \langle s_2, s_3\rangle$ is an abelian group of even order, $|s_3| \geqslant 3$, and $|A|/2 = |s_1| \geqslant |s_2| \geqslant |s_3|$, where at least one inequality is strict, then $\textrm{Cay}(A, \{s_1, s_2, s_3\})$ has a Hamilton decomposition.*

Corollary 4.1.6. *Let $\Delta = \textrm{Cay}(A/\langle s_3\rangle, \{\overline{s_1}, \overline{s_2}\})$ be a quotient of $\Gamma = \textrm{Cay}(A, \{s_1, s_2, s_3\})$ of order at least three. If $\overline{s_2}$ generates a Hamilton cycle in Δ, and $\langle s_1\rangle$ has index 2 in A, then Γ has a Hamilton decomposition.*

Corollary 4.1.7. *If $\Gamma = \textrm{Cay}(\mathbb{Z}_{2m}, \{a, b, c\})$ is connected, 6-regular, $|a| = m$, and $\gcd(2m, b, c) = 1$, then Γ has a Hamilton decomposition.*

In Chapter 5, combining the computational results of the Appendix, the following results are shown.

Theorem 5.4.1. *If $\Gamma = \textrm{Cay}(A, \{s_1, s_2, s_3\})$ is a connected, 6-regular, abelian Cayley graph of even order, and for some $1 \leqslant i \leqslant 3$, $\Delta_i = \textrm{Cay}(A/\langle s_i\rangle, \{\overline{s_{j_1}}, \overline{s_{j_2}}\})$ is 4-regular, and $\Delta_i \not\cong \textrm{Cay}(\mathbb{Z}_3, \{1, 1\})$, then Γ has a Hamilton decomposition.*

Alternatively, Theorem 5.4.1 may be stated as follows.

Theorem 5.4.2 *If $\Gamma = \textrm{Cay}(A, S)$ is a connected, 6-regular, abelian Cayley graph of even order, then Γ has a Hamilton decomposition if S has no involutions, and for some $s \in S$, $\textrm{Cay}(A/\langle s\rangle, \overline{S})$ is 4-regular, and of order at least 4.*

Corollary 5.4.3. *If $\Gamma = \textrm{Cay}(A, \{s_1, s_2, s_3\})$ is a connected, 6-regular, abelian Cayley graph of even order, then Γ has a Hamilton decomposition if one of the following holds:*

(a) $s_1 \in \langle s_2, s_3\rangle$, $s_2 \in \langle s_1, s_3\rangle$, and $[A : \langle s_3\rangle] \geqslant 4$, or

(b) $|s_1| \geqslant |s_2| > 2|s_3|$, or

(c) $|s_1| \geqslant |s_2| > |s_3|$, and either

  i. $|A| = (2k + 1)|s_3|$, with $k \geqslant 2$, or

  ii. $|A| \geqslant 4|s_3|$ and $|s_1|$ and $|s_2|$ are odd.

Chapter 2

Pseudo-Cartesian Products

2.1 The Pseudo-Cartesian Product of Cycles

Let $A_n = a_1 a_2 \cdots a_n a_1$ and $B_m = b_1 b_2 \cdots b_m b_1$ denote cycles of length n and m respectively, where all subscripts are expressed modulo n and m respectively. The r-pseudo-cartesian product of two cycles is a central tool in finding Hamilton decompositions.

Definition 2.1.1 (Liu [32])**.** For an integer $0 \leqslant r < m$, the *r-pseudo cartesian product* of A_n and B_m, denoted $A_n \,\Box_r\, B_m$, is the simple graph with vertex set $\{(a_i, b_j) : 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m\}$ and edge set consisting of horizontal and vertical edges.

Horizontal edges: $\{\{(a_i, b_j), (a_{i+1}, b_j)\}, \{(a_n, b_j), (a_1, b_{j+r})\} : 1 \leqslant i < n, 1 \leqslant j \leqslant m\}$

Vertical edges: $\{\{(a_i, b_j), (a_i, b_{j+1})\}, \{(a_i, b_1), (a_i, b_m)\} : 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m - 1\}$

The following is Remark 3.2 in [32]:

Remark 2.1.2 (Liu [32]). If $\gcd(r, m) = t$, in $A_n \,\Box_r\, B_m$, then the horizontal edges form a 2-factor H which consists of t cycles of length mn/t and any consecutive t rows of $A_n \,\Box_r\, B_m$ are on t different cycles of H. In other words, the horizontal edges in the b_i and b_j-rows are in the same cycle if and only if $i \equiv j \pmod t$. If H is given an orientation, so that each cycle in H becomes a directed cycle, then all horizontal edges in the rows contained in a particular cycle have the same direction.

The graph $A_n \,\Box_r\, B_m$ is embedded on a torus and drawn so that vertex (a_i, b_j) is in the *ith column* (i.e. $\{(a_i, b_j) : 1 \leqslant j \leqslant m\}$) and *$j$th row* ($\{(a_i, b_j) : 1 \leqslant i \leqslant n\}$). The parameter r is called the *jump number* for the graph because edges in the a_n-column jump down r rows (modulo m) to connect to vertices in the a_1-column. Call the b_j-row *even* if j is even, and *odd* if j is odd, where $1 \leqslant j \leqslant m$. Similarly, call the a_i-column *even* or *odd*, depending on the parity of i, where $1 \leqslant i \leqslant n$. An example is shown in Figure 2.1. The following establishes a well-known connection between r-pseudo cartesian products and connected 4-regular Cayley graphs. We first recall concepts from

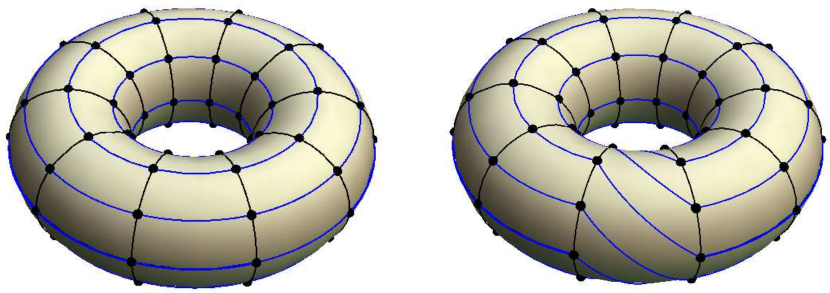


Figure 2.1: $A_{10} \,\square\, B_8$ (left) and $A_{10} \,\square_2\, B_8$ (right)

Section 2 of [9].

Theorem 2.1.3 (Bermond et al. [9])**.** *If* $\Gamma = \mathrm{C{\small AY}}(A, \{s_1, s_2\})$ *is a connected abelian Cayley graph,* $|s_1| \geqslant |s_2| = m \geqslant 3$*, and* $[A : \langle s_2 \rangle] = n \geqslant 3$*, then* $\Gamma \simeq A_n \,\square_r\, B_m$*, where* $ns_1 = rs_2$*.*

Proof. Without loss of generality, $|s_1| \geqslant |s_2| = m > 2$, $s_1 \neq \pm s_2$, and if $J := \langle s_2 \rangle$, then $\langle \overline{s_1} \rangle = A/J \Rightarrow |\overline{s_1}| = n > 2$. Hence, $ns_1 = rs_2$ for some $0 \leqslant r < m$. We claim $\varphi : V(A_n \,\square_r\, B_m) \to \Gamma$ is an isomorphism between Γ and $A_n \,\square_r\, B_m$, where

$$\varphi : (a_i, b_j) \mapsto (i-1)s_1 + (j-1)s_2,$$

and $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$. The vertical edges $\{(a_i, b_j), (a_i, b_{j+1})\} \in E(A_n \,\square_r\, B_m)$ are preserved, for

$$\{\varphi((a_i, b_j)), \varphi((a_i, b_{j+1}))\} = \{(i-1)s_1 + (j-1)s_2, (i-1)s_1 + js_2\} \in E(\Gamma).$$

Likewise, for $1 \leqslant i < n-1$ the horizontal edges $\{(a_i, b_j), (a_{i+1}, b_j)\}$ are preserved, for

$$\{\varphi((a_i, b_j)), \varphi((a_{i+1}, b_j))\} = \{(i-1)s_1 + (j-1)s_2, is_1 + js_2\} \in E(\Gamma).$$

Finally, the *jump-edges*, $\{(a_n, b_j), (a_1, b_{j+r})\}$ are also preserved because

$$\{\varphi((a_n, b_j)), \varphi((a_1, b_{j+r}))\} = \{(n-1)s_1 + (j-1)s_2, (j+r-1)s_2\},$$

and $(j+r-1)s_2 = ns_1 + (j-1)s_2 \Rightarrow ns_1 + (j-1)s_2 - ((n-1)s_1 + (j-1)s_2) = s_1$. The edges generated by s_1 form the horizontal 2-factor H in Remark 2.1.2, and so $t = \gcd(r, m) = [G : \langle s_1 \rangle]$. ■

Example 2.1.4. Let $\Gamma = \mathrm{C{\small AY}}(\mathbb{Z}_{30}, \{s_1 = 9, s_2 = 5\})$. As $\gcd(30, 5, 9) = 1$, $|s_2| = 30/\gcd(5, 30) = 6$, and $|s_1| = 30/\gcd(9, 30) = 10$, Γ is connected and 4-regular. Let $J = \langle 5 \rangle$. Now, $n = |\mathbb{Z}_{30} : J| = 5$ and r satisfies $5r = 5(9) = 45 \equiv 15 \pmod{30} \Rightarrow r = 3$. By Theorem 2.1.3, $\Gamma \simeq A_5 \,\square_3\, B_6$ where the isomorphism is $f : (a_i, b_j) \mapsto 9(i-1) + 5(j-1) \pmod{30}$, for $1 \leqslant i \leqslant 5$ and $1 \leqslant j \leqslant 6$. The two graphs are shown in Figure 2.2.

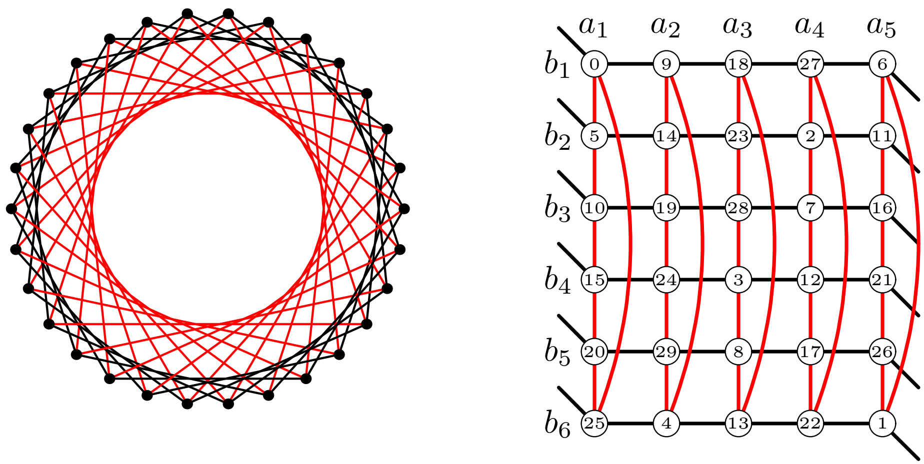


Figure 2.2: $\textsc{Cay}(\mathbb{Z}_{30},\ \{5,9\}) \simeq A_5 \,\square_3\, B_6$.

2.2 Edge Color-Switches

Definition 2.2.1. Color the vertical edges of $A_n \,\square_r\, B_m$ red, and the horizontal edges black. For integers i and j, where $1 \leqslant i < n$ and $1 \leqslant j \leqslant m$, we define an $\{a_i, a_{i+1}, b_j, b_{j+1}\}$-*color-switch* as an operation that interchanges the color of the edges

$$\{\{(a_i, b_j), (a_{i+1}, b_j)\}, \{(a_i, b_{j+1}), (a_{i+1}, b_{j+1})\}\}$$

with the color of the edges

$$\{\{(a_i, b_j), (a_i, b_{j+1})\}, \{(a_{i+1}, b_j), (a_{i+1}, b_{j+1})\}\}.$$

Similarly, for an integer $1 \leqslant j \leqslant m$, an $\{a_n, a_1, b_j, b_{j+1}\}$-*color-switch* interchanges the color of the edges

$$\{\{(a_n, b_j), (a_1, b_{j+r})\}, \{(a_n, b_{j+1}), (a_1, b_{j+1+r})\}\}$$

with the color of the edges

$$\{\{(a_n, b_j), (a_n, b_{j+1})\}, \{(a_1, b_{j+r}), (a_1, b_{j+1+r})\}\}.$$

For brevity, we shall denote this as $\{a_i, a_{i+1}, b_j, b_{j+1}\}$-CS. A *color-switching configuration*, or CS-*configuration*, is a set $\{X_i\}_{i=1}^{d}$, of color-switches which are edge-disjoint.

The following remark combines Facts 3.10 and 3.11 in [32]:

Remark 2.2.2 (Liu [32]). If the edges $\{x, y\}$ and $\{z, w\}$ lie on vertex-disjoint cycles of color c in $A_n \,\square_r\, B_m$, then applying an $\{a_i, a_{i+1}, b_j, b_{j+1}\}$-CS will join the two cycles into a single cycle of color c. If $\{x, y\}$ and $\{z, w\}$ lie on a cycle C, of length δ and color c, and are separated by at least two edges, then applying an $\{a_i, a_{i+1}, b_j, b_{j+1}\}$-CS will produce a cycle also having length δ and color c, if and only if, upon making C a directed cycle, C has x, y, z, w as a subsequence of vertices. See Figure 2.4.

CS-configurations have been used to obtain the following result.

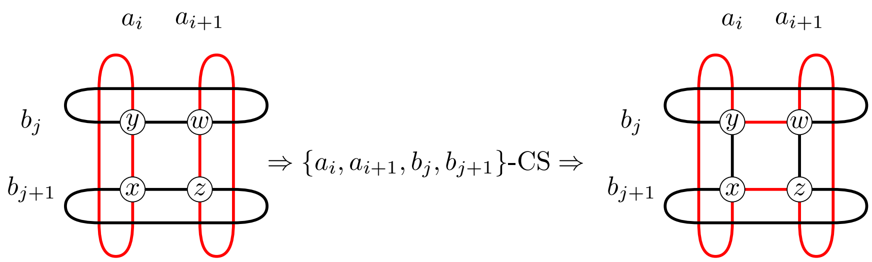


Figure 2.3: A red-black color-switch from Definition 2.2.1.

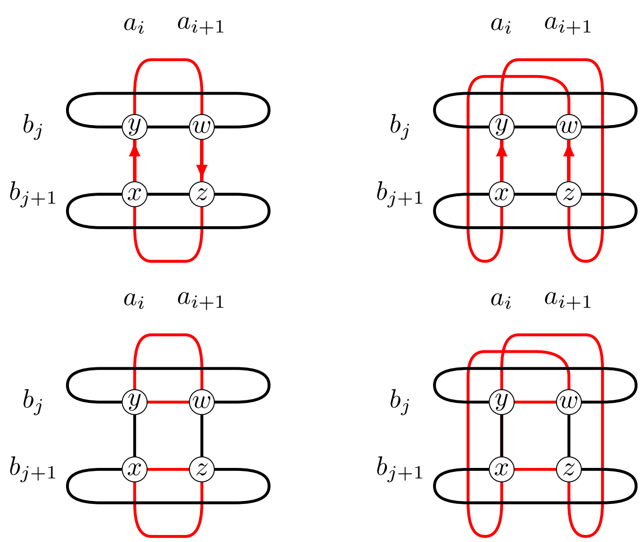


Figure 2.4: A red cycle-preserving color-switch from Remark 2.2.2.

Theorem 2.2.3 (Fan et al. [21])**.** *For all $n, m \geqslant 3$, then graph $A_n \,\square_r\, B_m$ has a Hamilton decomposition.*

Definition 2.2.4. For $d = 2d' \geqslant 2$, a *left-alternating horizontal switch*, denoted $\{a_i, a_{i+d}, b_j\}$-LAHS, is the CS-configuration consisting of the d color-switches,

$$\{\{a_{i+2x}, a_{i+1+2x}, b_j, b_{j+1}\}\text{-CS}, \{a_{i+1+2x}, a_{i+2+2x}, b_{j-1}, b_j\}\text{-CS} : 0 \leqslant x \leqslant (d-2)/2\}.$$

Likewise, a *right-alternating horizontal switch*, denoted $\{a_i, a_{i+d}, b_j\}$-RAHS, is the CS - configuration consisting of the d color-switches,

$$\{\{a_{i+2x}, a_{i+1+2x}, b_{j-1}, b_j\}\text{-CS}, \{a_{i+1+2x}, a_{i+2+2x}, b_j, b_{j+1}\}\text{-CS} : 0 \leqslant x \leqslant (d-2)/2\}.$$

Note that if the vertical edges in the a_j-columns for $i \leqslant j \leqslant i+d$ are in different cycles, then applying $\{a_i, a_{i+d}, b_j\}$-LAHS or $\{a_i, a_{i+d}, b_j\}$-RAHS will join all those cycles together, by Remark 2.2.2. Furthermore, if the horizontal edges in the b_k-rows where $j \leqslant k \leqslant j+1$ are in different cycles,

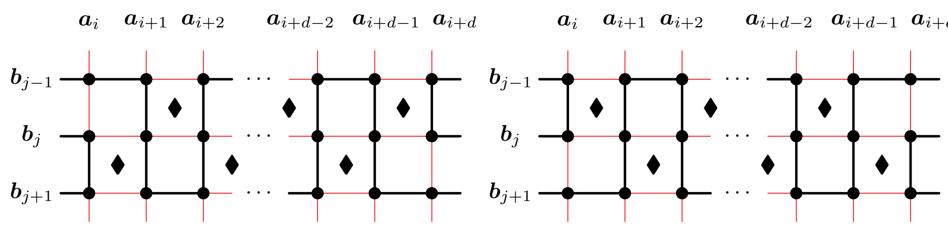


Figure 2.5: Left and right-alternating horizontal switches (LAHS and RAHS) of Definition 2.2.4.

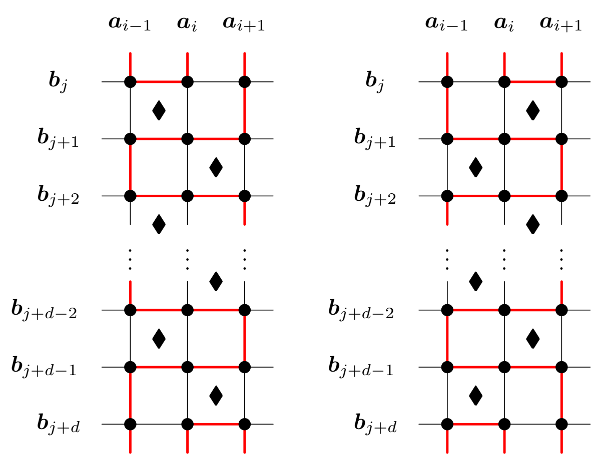


Figure 2.6: Left and right-alternating vertical switches (LAVS and RAVS) of Definition 2.2.5.

then a left or right-alternating switch will join the three horizontal cycles together.

Definition 2.2.5. If $d = 2d' \geqslant 2$, then a *left-alternating vertical switch*, denoted $\{a_i, b_j, b_{j+d}\}$-LAVS, is the CS-configuration consisting of the d color-switches,

$$\{\{a_{i-1}, a_i, b_{j+2x}, b_{j+1+2x}\}\text{-CS}, \{a_i, a_{i+1}, b_{j+1+2x}, b_{j+2+2x}\}\text{-CS} : 0 \leqslant x \leqslant (d-2)/2\}.$$

Likewise, a *right-alternating vertical switch*, denoted $\{a_i, b_j, b_{j+d}\}$-RAVS, is the CS - configuration consisting of the d color-switches,

$$\{\{a_i, a_{i+1}, b_{j+2x}, b_{j+1+2x}\}\text{-CS}, \{a_{i-1}, a_i, b_{j+1+2x}, b_{j+2+2x}\}\text{-CS} : 0 \leqslant x \leqslant (d-2)/2\}.$$

Definition 2.2.6. Changing a RAVS (or RAHS) to a LAVS (or LAHS) or vice versa, for a fixed set of parameters, will be called a *color-switch reflect*.

The remainder of this chapter outlines color-switching configurations that will be used throughout the rest of this thesis. Apply a 2-coloring to $A_n \square_r B_m$, with the vertical edges colored red, and the horizontal and jump edges colored black.

Theorem 2.2.7 (Liu [32, 34]). *Suppose there are $t = \gcd(r, m)$ horizontal cycles in $A_n \square_r B_m$ with the horizontal cycles colored c_1 and the vertical cycles colored c_2.*

(a) If $t = 2k + 1 \geqslant 3$, then applying a $\{a_i, b_{1+\ell}, b_{t+\ell}\}$-LAVS or RAVS to $A_n \square_r B_m$ for any $2 \leqslant i \leqslant n-1$ and any integer $0 \leqslant \ell \leqslant m-1$ will produce a Hamilton cycle C of color c_1, and join the vertical c_2-colored cycles in the a_{i-1}, a_i, and a_{i+1}-columns into one cycle C'. Furthermore, applying an

$$\{a_{i+1}, a_{i+2}, b_{t+\ell}, b_{t+1+\ell}\}\text{-}CS$$

preserves the Hamilton cycle and joins the a_{i+2}-column to C'.

(b) If $t = 2k \geqslant 4$, then applying a $\{a_i, b_{1+\ell}, b_{t-1+\ell}\}$-LAVS or RAVS and an

$$\{a_{i+1}, a_{i+2}, b_{t-1+\ell}, b_{t+\ell}\}\text{-}CS$$

to $A_n \square_r B_m$ for any $2 \leqslant i \leqslant n-2$ and any integer $0 \leqslant \ell \leqslant m-1$ will produce a Hamilton cycle of color c_1, and join the vertical c_2-colored cycles in the a_{i-1}, a_i, a_{i+1}, and a_{i+2}-columns into one cycle.

The justification for Theorem 2.2.7(a) is that upon applying an $\{a_i, b_1, b_t\}$-RAVS or -LAVS to $A_n \square_r B_m$, we may orient the resulting Hamilton cycle so that it becomes a directed cycle. Now, all edges in a particular b_j-row have the same direction, so we may speak of the *direction of the b_j-row*. After the alternating switching configuration, the b_j and b_{j+1}-rows have opposite direction for $1 \leqslant j \leqslant t-1$. As t is odd, the b_1 and b_t-rows have the same direction. Furthermore, the $b_1, b_{t+1}, b_{2t+1}, \ldots, b_{zt+1}$-rows all have the same direction, as they were in the same cycle initially. For the same reason, the $b_t, b_{2t}, \ldots, b_{jt}, \ldots, b_{kt} = b_m$-rows, and by extension, the b_1 and b_m-rows, have the same direction. By Remark 2.2.2, we may apply either an $\{a_{i+1}, a_{i+2}, b_t, b_{t+1}\}$-CS or an $\{a_{i+1}, a_{i+2}, b_m, b_1\}$-CS to preserve the Hamilton cycle. Thus, Theorem 2.2.7 generalizes as follows.

Theorem 2.2.8. *Suppose there are $t = \gcd(r, m)$ horizontal cycles in $A_n \square_r B_m$ colored c_1 and n vertical cycles colored c_2. Let $m = kt$, and i, ℓ, z, d be integers satisfying $2 \leqslant i < n$, $0 \leqslant \ell < m$, $i < z \leqslant n$, and $1 \leqslant d \leqslant k$.*

(a) If $t = 2k + 1 \geqslant 3$, then the application of an $\{a_i, b_{1+\ell}, b_{t+\ell}\}$-LAVS or RAVS to $A_n \square_r B_m$ will produce a Hamilton cycle, C, of color c_1, and join the vertical c_2-colored cycles in the a_{i-1}, a_i, and a_{i+1}-columns into one cycle C'. Furthermore, applying an

$$\{a_z, a_{z+1}, b_{dt+\ell}, b_{dt+1+\ell}\}\text{-CS}$$

will preserve C and connect the vertical edges in the a_z and a_{z+1}-columns into one cycle.

(b) If $t = 2k \geqslant 4$, the application of an $\{a_i, b_{1+\ell}, b_{t-1+\ell}\}$-LAVS or -RAVS to $A_n \square_r B_m$ and either an

$$\{a_z, a_{z+1}, b_{dt-1+\ell}, b_{dt+\ell}\}\text{-CS } \textit{or an } \{a_z, a_{z+1}, b_{m+\ell}, b_{1+\ell}\}\text{-CS}$$

will produce a Hamilton cycle of color c_1.

Lemma 2.2.9 (Liu [34]). *If there are $t = \gcd(r, m) = 2k \geqslant 2$ horizontal cycles in $A_n \square_r B_m$ colored c_1 and n vertical cycles colored c_2, then by switching the colors of the edge sets E_1 and E_2, where*

$$E_1 = \{(a_1, b_{2j-1})(a_1, b_{2j}) : 1 \leqslant j \leqslant m/2\} \cup \{(a_n, b_{2j})(a_n, b_{2j+1}) : 1 \leqslant j \leqslant m/2\},$$

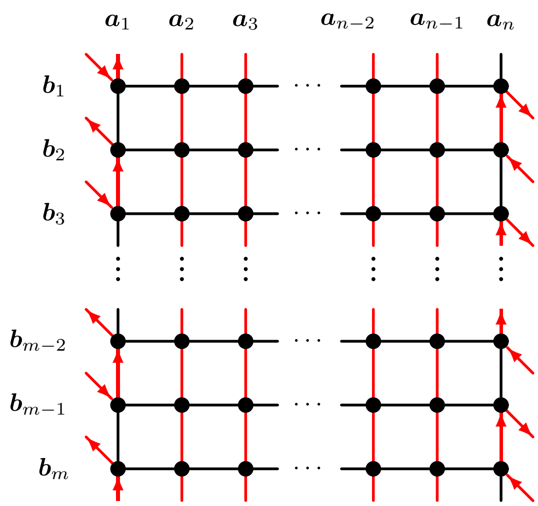


Figure 2.7: The CS-configuration of Lemma 2.2.9.

and $E_2 = \{(a_n, b_i)(a_1, b_{i+r}) : 1 \leqslant i \leqslant m\}$ *in* $A_n \square_r B_m$*, we obtain a Hamilton cycle of color* c_1 *and a cycle, colored* c_2*, consisting of the vertices in the* a_1 *and* a_n*-columns. Furthermore, if the* c_2*-colored cycle is given an orientation, all included vertical edges have the same direction. (See Figure 2.7.)*

The following simple observation is formulated as a lemma.

Lemma 2.2.10. *If the* CS*-configuration of Lemma 2.2.9 is applied to* $A_n \square_r B_m$*, for* t *even, then upon applying an* $\{a_i, a_{i+d}, b_{2j+1}\}$-RAHS*, for some* $d \geqslant 2$ *even, and* $j \geqslant 1$*, the* c_1*-colored Hamilton cycle is preserved.*

Lemma 2.2.11 (Liu [34])**.** *If* $n \geqslant 6$ *and* $t = \gcd(r, m) = 2k$*, apply an* $\{a_i, a_{i+1}, b_i, b_{i+1}\}$*-CS to* $A_n \square_r B_m$ *where* $i = 1$ *if* $k = 1$ *and* $i = 1, 2, 3$ *if* $k = 2$*. If* $k \geqslant 3$*, let* $i = 1, \ldots, 6$ *together with the* CS*-configuration*

$$\{\{a_1, a_2, b_j, b_{j+1}\}\text{-CS}, \{a_2, a_3, b_{j+1}, b_{j+2}\}\text{-CS}, \{a_3, a_4, b_{j+2}, b_{j+3}\}\text{-CS}, \{a_4, a_5, b_{j+3}, b_{j+4}\}\text{-CS}\},$$

where $j \equiv 2 \pmod 4$*, and*

(a) $6 \leqslant j \leqslant t-4$*, if* $t \equiv 2 \pmod 4$*, or*

(b) $6 \leqslant j \leqslant t-6$*, if* $t \equiv 0 \pmod 4$*. Additionally, apply an*

$$\{a_1, a_2, b_{t-2}, b_{t-1}\}\text{-CS } \textit{and an } \{a_2, a_3, b_{t-1}, b_t\}\text{-CS}.$$

The result is a black Hamilton cycle and a red cycle consisting of the vertical edges in the a_i*-columns where* $1 \leqslant i \leqslant d$ *and* $d = 2$ *if* $k = 1$*,* $d = 4$ *if* $k = 2$*, and* $d = 6$ *if* $k \geqslant 3$*.*

Definition 2.2.12. If $X := \{a_i, a_{i+1}, b_j, b_{j+1}\}$-CS is a fixed color-switch in $A_n \square_r B_m$, we say X is *c-incident* to the a_i and a_{i+1}-columns and *r-incident* to the b_j and b_{j+1}-rows. Furthermore, applying X an even number of times leaves the edge-colors invariant. Thus, *removing* a color-switch means applying X any even number of times.

Lemma 2.2.13. *If $n \geqslant 6$ and $t = \gcd(r, m) = 2k \geqslant 6$, then upon applying the* CS*-configuration of Lemma 2.2.11 to $A_n \Box_r B_m$, we may remove the $\{a_4, a_5, b_4, b_5\}$*-CS *and $\{a_5, a_6, b_5, b_6\}$*-CS*, apply an $\{a_4, a_5, b_5, b_6\}$*-CS *and an $\{a_5, a_6, b_4, b_5\}$*-CS*, and preserve the black Hamilton cycle and the red cycle on the a_i-columns, where $1 \leqslant i \leqslant 6$.*

Proof. As the order in which any set of color-switches are applied is invariant, we may view the new CS-configuration as having first applied an $\{\{a_i, a_{i+1}, b_i, b_{i+1}\}\text{-CS } : 1 \leqslant i \leqslant 3\}$, and then applying an $\{a_4, a_5, b_5, b_6\}$-CS and an $\{a_5, a_6, b_4, b_5\}$-CS to $A_n \Box_r B_m$. At this point, it is clear, by Remark 2.2.2, that a c_2-colored cycle is formed on the a_i-columns, where $1 \leqslant i \leqslant 6$. Each successive pair of color-switches as we move down the b_j-rows first breaks the c_2-colored cycle into two cycles, and then rejoins the two cycles again. By Remark 2.2.2, the result follows. ■

Definition 2.2.14 (Fan et al. [21])**.** Let $X_1, X_2, \ldots, X_k$, where $k \geqslant 3$ be a set of edge-disjoint color switches in $A_n \Box_r B_m$. Call $\{X_{k-1}, X_k\}$ a *good pair* if X_{k-2} is the right of X_i for all $i \leqslant k-3$, X_{k-1} is to the right of X_{k-2}, X_k is to the right of X_{k-1}, and there exists a positive integer y such that X_{k-2} and X_k are r-incident to the b_y and b_{y+1}-rows, and X_{k-1} is r-incident to either the b_y and b_{y-1}-rows or the b_y and b_{y+1}-rows.

Lemma 2.2.15 (Fan et al. [21])**.** *If $X_1, X_2, \ldots, X_k$, where $k \geqslant 3$ is a set of edge-disjoint color-switches in $A_n \Box_r B_m$, and $\{X_{k-1}, X_k\}$ form a good pair, then if after applying*

$$X_1, \ldots, X_{k-2},$$

there exists a c_1-colored Hamilton cycle, applying X_{k-1} and X_k will preserve this cycle.

The most common usage of Lemma 2.2.15 is the application of a LAHS or a RAHS, as these form color-switches that are pairwise, good pairs.

Corollary 2.2.16. *If a c_1-colored Hamilton cycle is created by applying a* CS*-configuration to $A_n \Box_r B_m$ such that, for some $1 \leqslant i < n$ and $1 \leqslant j \leqslant m$, $\{a_i, a_{i+1}, b_j, b_{j+1}\}$*-CS *is a rightmost color-switch, then applying an $\{a_{i+1}, a_{i+1+d}, b_j\}$*-RAHS *or an $\{a_{i+1}, a_{i+1+d}, b_{j+1}\}$*-LAHS*, for some $d \geqslant 2$ even, will preserve the Hamilton cycle. Furthermore, removing any two consecutive color-switches in the RAHS or LAHS will preserve the Hamilton cycle.*

The following type of color-switch will be used heavily in Chapter 4.

Definition 2.2.17. If Γ is a graph that contains $A_n \Box_r B_m$ as a spanning subgraph, and for some integers

$$C := (a_i, b_j)(a_i, b_{j+1})(a_k, b_{\ell+1})(a_k, b_\ell)(a_i, b_j)$$

is a 4-cycle of Γ with $(a_i, b_j)(a_i, b_{j+1})$ and $(a_k, b_\ell)(a_k, b_{\ell+1})$ colored c_2 and $(a_i, b_j)(a_k, b_\ell)$ and $(a_i, b_{j+1})(a_k, b_{\ell+1})$ colored c_1, then an $\{a_i, a_k, b_j, b_\ell\}$-*vertical oblique color-switch*, or -VOCS, is a color-switch that interchanges the colors of edges in C. Likewise, an $\{a_i, a_k, b_j, b_\ell\}$-*horizontal oblique color-switch*, or -HOCS, in $A_n \Box_r B_m$ is a color-switch that interchanges the colors of edges in the 4-cycle,

$$(a_i, b_j)(a_{i+1}, b_j)(a_k, b_\ell)(a_{k+1}, b_\ell)(a_i, b_j),$$

where $1 \leqslant i \neq k \leqslant n$ and $1 \leqslant j \neq \ell \leqslant m$.

Note, if $k = i + 1$ and $\ell = j + 1$, in Definition 2.2.17, an $\{a_i, a_k, b_j, b_\ell\}$-VOCS or -HOCS is an $\{a_i, a_{i+1}, b_j, b_{j+1}\}$-CS.

Chapter 3

Hamilton Decompositions for Graphs of Odd Order

3.1 Lifting to the 6-Regular Case

This chapter offers a proof of Alspach's conjecture for the odd order, 6-regular case, hence generalizing Theorem 1.5.2. A proof of this result was published by Westlund, Liu, Kreher [53] in 2009, though we re-present the proof (using more unifying notation) because the technique used will help to understand many of the constructions in Chapter 5. In this section, preliminary theorems and definitions from [21, 32, 33, 34] are presented, and where needed, proofs are supplied for clarity. We first recall the fundamental definition of a quotient graph.

Definition 3.1.1. If $\Gamma = \textsc{Cay}(A, S)$ has connection set $S = \{s_1, \ldots, s_k\}$, and $J := \langle s_k \rangle$, then the Cayley graph $\Delta = \textsc{Cay}(A/J, \overline{S})$, where $\overline{S} = \{\overline{s_i} : i = 1, \ldots, k-1\}$ and $\overline{s_i} = s_i + J$, is called a *quotient graph of* Γ. If Γ is connected, then Δ is connected.

Definition 3.1.2. If $\{\overline{x}, \overline{y}\} \in E(\Delta)$, where $\overline{x} - \overline{y} = \overline{s_i}$, then the set

$$L_\Delta\{\overline{x}, \overline{y}\} = \{\{u, v\} : \overline{u} = \overline{x}, \overline{v} = \overline{y}, u - v = s_i\}$$

is called the *lift* $\{\overline{x}, \overline{y}\}$. Furthermore, given a subgraph $\overline{F}$ of Δ, let F be the subgraph of Γ that is induced on the lifts of all edges of $\overline{F}$. Then F is called the *lift of the subgraph* $\overline{F}$, or we say F *is the subgraph that* $\overline{F}$ *lifts to.*

Edge-disjoint subgraphs of Δ lift to edge-disjoint subgraphs of Γ. Lifting Hamilton cycles in quotient graphs is by no means a new idea, and many approaches can be taken, e.g., Alspach [3]. For the benefit of the reader, we now present proofs of the following two important results, which are Lemma 3.7 and Corollary 3.9 in [32], respectively.

Lemma 3.1.3 (Liu [32]). *If $\Gamma = \text{Cay}(A, \{s_1, s_2, s_3\})$ is a connected Cayley graph on an abelian group A, then any Hamilton cycle $\overline{H}$ of $\Delta = \text{Cay}(A/\langle s_3\rangle, \{\overline{s_1}, \overline{s_2}\})$ lifts to a 2-factor H of Γ. Furthermore, the union of H and the 2-factor generated by s_3 is an r-pseudo-cartesian product.*

Proof. Let $|s_3| = m$, $n = |A : \langle s_3\rangle|$, $A/\langle s_3\rangle = \{\overline{g_1}, \ldots, \overline{g_n}\}$, where $\overline{g_1} = \overline{0}$. Suppose that $\overline{H}$ is a Hamilton cycle of Δ, where

$$\overline{H} = \overline{g_{\pi(1)}}, \overline{g_{\pi(2)}}, \ldots, \overline{g_{\pi(n)}}, \overline{g_{\pi(1)}}, \text{ for some } \pi \in \text{Sym}(n).$$

For all $1 \leqslant k \leqslant n$, without loss of generality, $\overline{g_{\pi(k)}} - \overline{g_{\pi(k+1)}} = \overline{s_j}$, for some $j \in \{1, 2\}$. Thus, for $1 \leqslant k \leqslant n-1$, there exist integers x_k and y_k such that

$$(g_{\pi(k)} + x_k s_3) - (g_{\pi(k+1)} + y_k s_3) = s_j.$$

Let $a_{\pi(k)} := g_{\pi(k)} + x_k s_3$ and $a_{\pi(k+1)} := g_{\pi(k+1)} + y_k s_3$. Hence,

$$L_\Delta\{\overline{g_{\pi(k)}}, \overline{g_{\pi(k+1)}}\} = \{\{a_{\pi(k)} + \ell s_3, a_{\pi(k+1)} + \ell s_3\} : 0 \leqslant \ell < m\}.$$

Next, as $\overline{g_{\pi(1)}} = \overline{a_{\pi(1)}}$ and $\overline{g_{\pi(n)}} = \overline{a_{\pi(n)}}$, there exist integers q_1 and q_2, for which

$$(a_{\pi(n)} + q_1 s_3) - (a_{\pi(1)} + q_2 s_3) = s_j \Rightarrow a_{\pi(n)} - (a_{\pi(1)} + (q_2 - q_1)s_3) = s_j.$$

Let $r = q_2 - q_1$, so that $\{a_{\pi(n)}, a_{\pi(1)} + rs_3\} \in E(\Gamma)$. Hence,

$$L_\Delta\{\overline{g_{\pi(n)}}, \overline{g_{\pi(1)}}\} = \{\{a_{\pi(n)} + \ell s_3, a_{\pi(1)} + (r + \ell)s_3\} : 0 \leqslant \ell < m\}.$$

Thus, a path of length n is formed:

$$P_n^j = a_{\pi(1)} + jrs_3, a_{\pi(2)} + jrs_3, \ldots, a_{\pi(n)} + jrs_3, a_{\pi(1)} + (j+1)rs_3,$$

and the union of these paths

$$\bigcup_{j=0}^{d-1} P_n^j,$$

where $d-1$ is the integer such that $drs_3 = 0$, forms a cycle of length dn. Hence, $dr = \text{lcm}(m, r)$ and so $d = m/\gcd(r, m)$. It follows that the lift of $\overline{H}$, denoted H, is a 2-factor of Γ, where

$$H = \bigcup_{j=1}^{n} L_\Delta\{\overline{g_{\pi(j)}}, \overline{g_{\pi(j+1)}}\}.$$

Futhermore, $F_j := a_{\pi_i(j)}, a_{\pi_i(j)} + s_3, \ldots, a_{\pi_i(j)} + (m-1)s_3, a_{\pi_i(j)}$ is a cycle for each $j = 1, 2, \ldots, n$. Thus, $F = \bigcup_{j=1}^{n} F_j$ is 2-factor of Γ. Viewing the edges of H as horizontal, and the edges of F as vertical, let $\theta : V(H \cup F) \to V(A_n \,\square_r\, B_m)$ be defined by

$$\theta : a_i + (j-1)s_3 \mapsto (a_i, b_j) \text{ for } 1 \leqslant i \leqslant n \text{ and } 1 \leqslant j \leqslant m.$$

Clearly, $H \cup F = A_n \,\square_r\, B_m$, where $A_n = a_{\pi(1)} a_{\pi(2)} \cdots a_{\pi(n)} a_{\pi(1)}$. ∎

We now recall an important class of graphs, established in [32], and discuss their relationship with quotient graphs of Cayley graphs.

Definition 3.1.4 (Liu [32])**.** For $m, n \geqslant 3$, let a $D(3,m,n)$-*graph* be a 6-regular graph $G = (V, E)$ satisfying:

1. $V(G) = \{(a_i, b_j) : 1 \leqslant i \leqslant n \text{ and } 1 \leqslant j \leqslant m\}$, and

2. $E(G)$ can be partitioned into the three sets F, H_1, and H_2, where

$$\begin{aligned}
F &= \{\{(a_i,b_j),(a_i,b_{j+1})\},\{(a_i,b_1),(a_i,b_m)\} : 1 \leqslant i \leqslant n, 1 \leqslant j < m\},\\
H_1 &= \{\{(a_{\pi_1(i)},b_j),(a_{\pi_1(i+1)},b_j)\},\{(a_{\pi_1(n)},b_j),(a_{\pi_1(1)},b_{j+r_1})\} : 1 \leqslant i < n, 1 \leqslant j \leqslant m\},\\
H_2 &= \{\{(a^2_{\pi_2(i)},b_j),(a^2_{\pi_2(i+1)},b_j)\},\{(a^2_{\pi_2(n)},b_j),(a^2_{\pi_2(1)},b_{j+r_2})\} : 1 \leqslant i < n, 1 \leqslant j \leqslant m\},
\end{aligned}$$

where r_1 and r_2 are integers, and for $k = 1, 2$, $0 \leqslant r_k < m$, $\pi_k \in \mathrm{SYM}(n)$, and $(a^2_t, b_j) = (a_t, b_{j+h_t})$ for some integer h_t, where $0 \leqslant h_t < m$.

Remark 3.1.5 (Liu [32])*.* $D(3,m,n)$-graphs can be viewed as two pseudo-cartesian products that share a common vertical 2-factor. Indeed, F, H_1, and H_2 are each 2-factors, $H_1 \cup F \simeq A^{(1)}_n \square_{r_1} B_m$, and $H_2 \cup F \simeq A^{(2)}_n \square_{r_2} B_m$, with the edges of F vertical, the edges of H_j horizontal, and for $k = 1, 2$,

$$A^{(k)}_n := a^2_{\pi_k(1)} a^2_{\pi_k(2)} \cdots a^2_{\pi_k(n)} a^2_{\pi_k(1)}.$$

When the labeling is defined so that $\pi_1 = (1)$, we will write A_n in place of $A^{(1)}_n$. For the remainder of this thesis, whenever a $D(3,m,n)$-graph is discussed, the edges in F will be colored red, the edges in H_1 colored blue, and the edges in H_2 colored black (e.g., see Figure 3.1).

Theorem 3.1.6 (Liu [32])**.** *If* $\Gamma = \mathrm{CAY}(A, \{s_1, s_2, s_3\})$ *and if* $\Delta = \mathrm{CAY}(A/\langle s_3\rangle, \{\overline{s_1}, \overline{s_2}\})$ *can be decomposed into two Hamilton cycles,*

$$\overline{H_1} := \overline{a_{\pi_1(1)}}, \overline{a_{\pi_1(2)}}, \ldots, \overline{a_{\pi_1(n)}}, \overline{a_{\pi_1(1)}} \quad \text{and} \quad \overline{H_2} := \overline{a_{\pi_2(1)}}, \overline{a_{\pi_2(2)}}, \ldots, \overline{a_{\pi_2(n)}}, \overline{a_{\pi_2(1)}},$$

then Γ *is a* $D(3,m,n)$*-graph, with* $m = |s_3|$, $n = [A : \langle s_3\rangle]$, *where* H_i *is the 2-factor that* $\overline{H_i}$ *lifts to, and* F *the 2-factor generated by* s_3*.*

Proof. By Lemma 3.1.3, we have $H_1 \cup F \simeq A_n \square_{r_1} B_m$, and without loss of generality $\pi_i = (1)$, so $A_n := a_1 a_2 \cdots a_n a_1$. By the same lemma, $H_2 \cup F \simeq A^{(2)}_n \square_{r_2} B_m$ where

$$A^{(2)}_n := a^2_{\pi_2(1)} a^2_{\pi_2(2)} \cdots a^2_{\pi_2(n)} a^2_{\pi_2(1)},$$

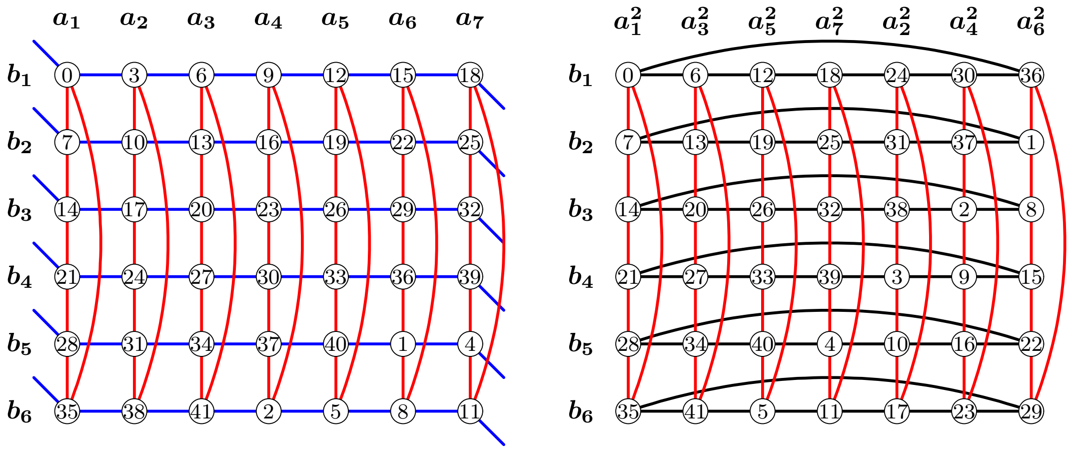
where (a^2_i, b_j) is the vertex $a^2_i + (j-1)s_3$ in Γ. Now, $\overline{a^2_i} = \overline{a_i}$, and so $a^2_i = a_i + h_i s_3$. Therefore,

$$(a^2_i, b_j) \leftrightarrow a^2_i + (j-1)s_3 = a_i + h_i s_3 + (j-1)s_3 = a_i + (h_i + j - 1)s_3 \leftrightarrow (a_i, b_{j+h_i}).$$

Hence, Γ is a $D(3,m,n)$-graph, and the a^2_i-column of $A^{(2)}_n \square_{r_2} B_m$ is just a cyclic shift of the elements in the a_i-column of $A_n \square_{r_1} B_m$. ∎

Example 3.1.7. Consider the circulant graph $\Gamma = \mathrm{CAY}(\mathbb{Z}_{42}, \{s_1 = 3, s_2 = 6, s_3 = 7\})$ (see Figure 3.1). If $J = \langle 7\rangle$, then the quotient graph is $\Delta = \mathrm{CAY}(\mathbb{Z}_{42}/\langle 7\rangle, \{\overline{3}, \overline{6}\})$. Δ is Hamilton decomposable into $\overline{H_1}$ and $\overline{H_2}$, where $\pi_1 = (1)$ and $\pi_2 = (235)(476)$, so that

$$\overline{H_1} = \overline{a_1}, \overline{a_2}, \overline{a_3}, \overline{a_4}, \overline{a_5}, \overline{a_6}, \overline{a_7}, \overline{a_1} = \overline{0}, \overline{3}, \overline{6}, \overline{2}, \overline{5}, \overline{1}, \overline{4}, \overline{0},$$



$H_1 \cup F = A_7 \,\square_3\, B_6$ $\qquad$ $H_2 \cup F = A_7^{(2)} \,\square\, B_6$

Figure 3.1: The graph $\textsc{Cay}(\mathbb{Z}_{42}, \{3,6,7\})$ from Example 3.1.7 is a $D(3,6,7)$-graph.

and

$$\overline{H_2} = \overline{a_1}, \overline{a_3}, \overline{a_5}, \overline{a_7}, \overline{a_2}, \overline{a_4}, \overline{a_6}, \overline{a_1} = \overline{0}, \overline{6}, \overline{5}, \overline{4}, \overline{3}, \overline{2}, \overline{1}, \overline{0}.$$

Thus, by Theorem 3.1.6, Γ is a $D(3,6,7)$-graph. F, H_1, and H_2 are the 2-factors generated by 7, 3 and 6, respectively, and $H_1 \cup F = A_7 \,\square_3\, B_6$ and $H_2 \cup F = A_7^{(2)} \,\square\, B_6$. Also, $(a_i^2, b_t) = (a_i, b_{t+h_i})$, where $h_i = 0$ for $i \in \{1,3,5,7\}$ and $h_i = 3$ for $i \in \{2,4,6\}$.

Remark 3.1.8. For brevity, we will write $\{a_i^2, b_j, b_{j+d}\}$-RAVS or LAVS to mean an alternating vertical switch between the $a^2_{\pi(k-1)}$, $a^2_{\pi(k)}$, and $a^2_{\pi(k+1)}$-columns, where $\pi(k) = i$. When we say *apply an* $\{a_i, a_{i+1}, b_j, b_{j+1}\}$-CS, it is assumed the switch is applied to $H_1 \cup F \simeq A_n \,\square_{r_1}\, B_m$. When we say *apply an* $\{a_i^2, a_{i+1}^2, b_j, b_{j+1}\}$-CS, it is assumed the switch is applied to $H_2 \cup F \simeq A_n^{(2)} \,\square_{r_2}\, B_m$.

The following two results are fundamental in proving Theorem 3.2.1.

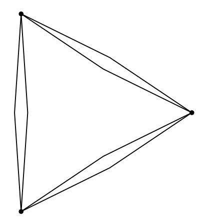
Theorem 3.1.9 (Fan et al. [21])**.** *Let A be a finite abelian group, and $S = \{s_1, s_2, s_3\}$ be a generating set for A. If S has an element s_i of odd order such that $\langle s_i \rangle$ is a subgroup of index at least nine, and $|\overline{s}_j| \geqslant 3$ for $j \neq i$, then $\textsc{Cay}(A,S)$ has a Hamilton decomposition.*

Theorem 3.1.10 (Fan et al. [21])**.** *Let A be a finite abelian group of odd order, with generating set $S = \{s_1, s_2, s_3\}$. If there exists an element of strictly smallest order, then $\textsc{Cay}(A,S)$ has a Hamilton decomposition.*

3.2 Decompositions for Odd Order Groups

Note, if $\Gamma = \textsc{Cay}(A,S)$ is connected, 6-regular, and $|A|$ is odd, then $S = \{s_1, s_2, s_3\}$, and $|s_i| \geqslant 3$ for $i = 1,2,3$.

Theorem 3.2.1 ([53])**.** *Every connected 6-regular Cayley graph on an abelian group of odd order has a Hamilton decomposition.*



$\Lambda_1 := \textsc{Cay}(\mathbb{Z}_3, \{\pm 1\})$

Figure 3.2: The quotient graph Δ of Case 1.i. of Theorem 3.2.1.

Proof. Suppose $\Gamma = \textsc{Cay}(A, S)$ is a connected 6-regular Cayley graph on a finite abelian group A of odd order. Then, without loss of generality, we may assume that the generating set is $S = \{s_1, s_2, s_3\}$ where $|s_1| \geqslant |s_2| \geqslant |s_3|$. Furthermore by Lagrange's Theorem, $|s_i| \geqslant 3$ is odd, for $i = 1, 2, 3$. The cases where $|s_i| = |A|$ for some $i = 1, 2, 3$, $|s_2| > |s_3|$, or S is a minimal generating set, are completely solved by Theorems 1.5.2, 3.1.10, or 1.5.3, respectively. Thus we may assume $|s_2| = |s_3|$. If $s_1 \in \langle s_3 \rangle$ and $s_2 \in \langle s_3 \rangle$, then $\langle s_3 \rangle = A$, i.e. Γ is a circulant, which is resolved by Theorem 1.5.2. If $s_1 \in \langle s_3 \rangle$, but $s_2 \notin \langle s_3 \rangle$, then $|s_1| = |s_2| = |s_3|$, and so without loss of generality, there are two cases to consider. Either $s_1 \notin \langle s_3 \rangle$ and $s_2 \notin \langle s_3 \rangle$ or $s_1 \notin \langle s_3 \rangle$ and $s_2 \in \langle s_3 \rangle$.

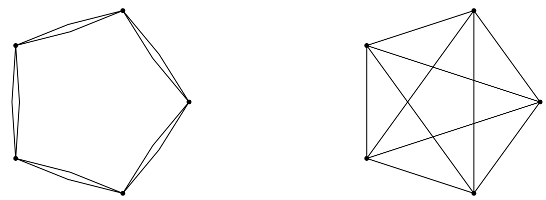
Case 1: $s_1 \notin \langle s_3 \rangle$ and $s_2 \notin \langle s_3 \rangle$

Let $J = \langle s_3 \rangle$, $m = |J|$, and $n = |A/J|$. Thus, $|A| = nm$, and by Lagrange's Theorem, both m and n are odd. Let $\overline{S} := \{\overline{s_1}, \overline{s_2}\}$. Then, as $\overline{s_i} \neq \overline{0}$ and n is odd, we have that $|\overline{s_i}| \geqslant 3$ is odd, for $i = 1, 2$. The quotient graph $\Delta := \textsc{Cay}(A/J, \overline{S})$ is a 4-regular connected Cayley graph. As $J \neq A$, and the case $n \geqslant 9$ is settled by Theorem 3.1.9, we may assume $3 \leqslant n \leqslant 7$. As n is prime, $|\overline{s_1}| = |\overline{s_2}| = n$, and $A/J \cong \mathbb{Z}_n$. Then, letting $a_i := (i-1)s_1$, where $1 \leqslant i \leqslant n$, we have:

$$\overline{H_1} := \overline{a_1}, \overline{a_2}, \ldots, \overline{a_n}, \overline{a_1} \quad \text{and} \quad \overline{H_2} := \overline{a_{\pi(1)}}, \overline{a_{\pi(2)}}, \ldots, \overline{a_{\pi(n)}}, \overline{a_{\pi(1)}}$$

is a Hamilton decomposition of Δ, where $\overline{H_i}$ is generated by $\overline{s_i}$, and π is a permutation of $\{1, 2, \ldots, n\}$. The cycles $\overline{H_1}$ and $\overline{H_2}$ lift to two 2-factors, H_1 and H_2, which are generated by s_1 and s_2, respectively. H_1 consists of t cycles of length $mn/t \geqslant m$, and H_2 consists of n cycles of length m. If $t = 1$, then Γ is a circulant, and we may apply Theorem 1.5.2. Hence we may assume that $3 \leqslant t \leqslant n$ and that t is odd. If $3 \leqslant m \leqslant 7$, then $|A|$ is the product of two odd primes, and we can apply Theorem 1.5.5. Thus, we may further assume that $nm > |s_1| \geqslant |s_2| = m \geqslant 9$. By Theorem 3.1.6, Γ is a $D(3, m, n)$-graph. Color the edges of F red, the edges of H_1 black, and the edges of H_2 blue. Hence, $H_1 \cup F \simeq A_n \square_{r_1} B_m$ and $H_2 \cup F \simeq A_n^2 \square_{r_2} B_m$. Also, by Remark 2.1.2, $t = \gcd(r_1, m)$ and $n = \gcd(r_2, m)$, so that $m = (2k+1)n$ for some $k > 0$. If $r_i = 0$, then $|s_i| = n < m$, a contradiction. Thus $r_i \neq 0$ for $i = 1, 2$.

i. If $n = 3$, then $t = 3$, and any three consecutive rows in $A_3 \square_{r_1} B_m$, respectively $A_3^{(2)} \square_{r_2} B_m$, lie on three different cycles. As $\overline{H_1} = \overline{H_2}$, up to a relabeling of the vertices, $\Delta = \Lambda_1 := \textsc{Cay}(\mathbb{Z}_3, \{\pm 1\})$, a fat triangle, shown Figure 3.2. By Remark 2.2.2, applying an $\{a_1, a_3, b_2\}$-RAHS to $C_3 \square_{r_1} B_m$, will join the three red cycles of F, respectively the three black cycles of H_1, into Hamilton cycles. As $m \geqslant 9$, there exists an integer d such that all vertical edges between the b_j-rows for $d \leqslant j \leqslant d+2$ in $A_3^{(2)} \square_{r_2} B_m$ are red. Then, a Hamilton decomposition is obtained by applying an $\{a_1^2, a_3^2, b_{d+1}\}$-RAHS.



$\Lambda_1 := \textsc{Cay}(\mathbb{Z}_5, \{\pm 1\})$ $\qquad$ $\Lambda_2 := \textsc{Cay}(\mathbb{Z}_5, \{1, 2\})$

Figure 3.3: The quotient graphs of Case 1.ii. of Theorem 3.2.1 and Case 1 of Lemma 5.3.1.

ii. If $n = 5$, then $t \in \{3, 5\}$. If $t = 5$ (i.e. $|s_i| = m$ for $1 \leqslant i \leqslant 3$), then by Remark 2.2.2, the application of an $\{a_2, b_1, b_5\}$-LAVS to $A_5 \,\square_{r_1}\, B_m$ yields a black Hamilton cycle, and joins together 3 of the 5 red cycles of F. Note, up to a relabeling of the vertices, $\Delta = \Lambda_1 := \textsc{Cay}(\mathbb{Z}_5, \{\pm 1\})$, a fat 5-cycle, or $\Delta = \Lambda_2 := \textsc{Cay}(\mathbb{Z}_5, \{1, 2\})$, a complete graph, shown in Figure 3.3.

If $\Delta = \Lambda_1$, then $\overline{H_1} = \overline{H_2}$, and without loss of generality, $\pi = (1)$. The red and black vertical edges in the a_3-column form a matching. Let d be any integer such that $e = (a_3^2, b_{1+d})(a_3^2, b_{2+d})$ is a red edge. Apply an $\{a_4^2, b_{1+d}, b_{5+d}\}$-LAVS to $A_5^{(2)} \,\square_{r_2}\, B_m$, to obtain a set of three monochromatic Hamilton cycles in Γ.

If $\Delta = \Lambda_2$, without loss of generality, $\pi = (2354)$, so that

$$A_5^{(2)} = a_1^2\, a_3^2\, a_5^2\, a_2^2\, a_4^2\, a_1^2.$$

Let h be the integer such that all vertical edges in the a_2^2-column that are between the b_{j+h}-rows, where $1 \leqslant j \leqslant 5$, are black. Now, as $5 \mid m \Rightarrow m \geqslant 15$, and so

$$(a_2^2, b_{5+h}), (a_2^2, b_{6+h}), (a_2^2, b_{7+h}), (a_2^2, b_{8+h})$$

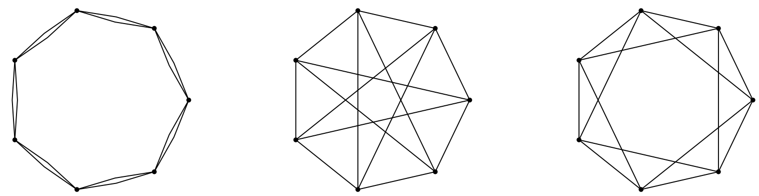
is a red 5-path. All vertical edges in the a_4^2- and a_5^2-columns are red, and so applying an $\{a_2^2, b_{5+h}, b_{9+h}\}$-LAVS or -RAVS to $H_2 \cup F$ yields 3 monochromatic Hamilton cycles. In the case $t = 3$, apply an $\{a_1, a_2, b_2\}$-LAVS to $H_1 \cup F$, and use the aforementioned technique to obtain the result.

iii. If $n = 7$, then $t \in \{3, 5, 7\}$. If $t = 7$, then $|s_i| = m \geqslant 21$ for $i = 1, 2, 3$. By Remark 2.2.2, applying an $\{a_2, b_1, b_7\}$-LAVS to $A_7 \,\square_{r_1}\, B_m$ produces a black Hamilton cycle, and joins all vertical red edges in the a_i-columns, where $i = 1, 2, 3$, to one cycle. Similar to the previous cases, up to a relabeling of the vertices, the quotient graph Δ may be viewed as one of the following graphs shown in Figure 3.4.

If $\Delta = \Lambda_1$, then without loss of generality, $\pi = (1)$, and apply an $\{a_3, a_5, b_2\}$-RAHS to $A_7 \,\square_{r_1}\, B_m$ to join two more of the red cycles of F into one red $5m$-cycle. By Lemma 2.2.15, this preserves the black cycle. Let d be any integer such that $e = (a_5^2, b_{1+d})(e_5^2, b_{2+d})$ is a red edge and apply an $\{a_6^2, b_{1+d}, b_{7+d}\}$-LAVS to $A_7^{(2)} \,\square_{r_2}\, B_m$ to obtain a Hamilton decomposition.

If $\Delta = \Lambda_2$, then without loss of generality, $\pi = (243756)$, so that

$$A_7^{(2)} = a_1^2\, a_4^2\, a_7^2\, a_3^2\, a_6^2\, a_2^2\, a_5^2\, a_1^2.$$



$\Lambda_1 := \textsc{Cay}(\mathbb{Z}_7, \{\pm 1\})$ $\Lambda_2 := \textsc{Cay}(\mathbb{Z}_7, \{1,3\})$ $\Lambda_3 := \textsc{Cay}(\mathbb{Z}_7, \{1,2\})$

Figure 3.4: The quotient graphs of Case 1.iii of Theorem 3.2.1 and Case 2 of Lemma 5.3.1.

Apply an $\{a_3, a_5, b_2\}$-RAHS to $A_7 \square_{r_1} B_m$ to join all red edges in the a_i-columns into one monochromatic cycle, where $1 \leqslant i \leqslant 5$. Let h be any integer such that the vertical edges in the a_3^2-column and between the b_{j+h}-rows, $1 \leqslant j \leqslant 7$ contain all four black edges. Clearly, $\{(a_3^2, b_{i+h}) : 7 \leqslant i \leqslant 13\}$ is a red 6-path. Thus, apply an $\{a_3^2, b_{7+h}, b_{13+h}\}$-LAVS or -RAVS to obtain a Hamilton decomposition. The case where $t = 3$ or $t = 5$ follow similarly.

If $\Delta = \Lambda_3$, then without loss of generality, $\pi = (235)(476)$, so that

$$A_7^2 = a_1^2\, a_3^2\, a_5^2\, a_7^2\, a_2^2\, a_4^2\, a_6^2\, a_1^2.$$

Let h be an integer such that all vertical edges in the a_2^2-column that are between the b_{j+h}-rows, where $1 \leqslant j \leqslant 7$, are black. Apply an $\{a_5^2, b_{3+h}, b_{7+h}\}$-LAVS or -RAVS, depending on if $(a_3^2, b_{3+h})(a_3^2, b_{4+h})$ is red or not, to join 5 of the 7 blue cycles of H_2 into one cycle and join the red edges in the a_i-columns into one cycle where $i = 1, 2, 3, 5, 5, 7$. By Remark 2.2.2, the application of an $\{a_2^2, a_6^2, b_{8+h}\}$-RAHS or -LAHS to produce a Hamilton decomposition.

Case 2: $s_1 \notin \langle s_3 \rangle$ and $s_2 \in \langle s_3 \rangle$.

In this case, $\langle s_2 \rangle = \langle s_3 \rangle$ and $s_2, s_3 \notin \langle s_1 \rangle$, for otherwise Γ is a circulant graph. Hence, we apply the technique of Case 1, by setting $J := \langle s_1 \rangle$. ∎

Chapter 4

A Decomposition for Non-Minimal Connection Sets

4.1 Using a Subgroup of Index 2

In this chapter, we obtain results when the connection set $S = \{s_1, s_2, s_3\}$ contains elements that have a linear dependency among themselves, i.e., S is not a minimal set of generators, and one element generates a subgroup of index two. The following observation will be used in the proof of Theorem 4.1.5.

Remark 4.1.1. If $\text{Cay}(A, \{s_1, s_2\}) \simeq A_n \,\square_r\, B_m$, and $x = (p-1)s_1 + (q-1)s_2 \leftrightarrow (a_p, b_q)$, then $-x$, the inverse of x, corresponds to the vertex (a_{p^*}, b_{q^*}) in $A_n \,\square_r\, B_m$, where $p^* = n - p + 2$ and $q^* = m - (q + r - 2)$. This is because

$$\varphi : (a_{p^*}, b_{q^*}) \mapsto (n - p + 1)s_1 + (m - q - r + 1)s_2,$$

and as $ns_1 = rs_2$ and $ms_2 = 0$, the above simplifies to $(1 - p)s_1 + (1 - q)s_2 = -x$.

Definition 4.1.2. If a CS-configuration has been applied to $A_n \,\square_r\, C_m$ to create a Hamilton cycle of (red) vertical edges, and upon making this cycle a directed cycle, we find that all vertical edges in the a_i-column, for some $1 \leqslant i \leqslant n$, have the same direction (either $\uparrow$ or $\downarrow$), we say the *a_i-column direction* is $\uparrow$ or $\downarrow$, respectively. More generally, we say $A_n \,\square\, B_m$ has *column-direction pattern* $\uparrow \downarrow \uparrow \downarrow \cdots$ if neighboring columns have different directions (depending on the parity of n, the a_1 and a_n-columns may have the same direction or not).

Lemma 4.1.3. *If $\Gamma = \textsc{Cay}(A, \{s_1, s_2, s_3\})$ is a 6-regular, connected Cayley graph of even order, satisfying*

$$2h + 1 = |A : \langle s_3\rangle| \geqslant |A : \langle s_2\rangle| \geqslant |A : \langle s_1\rangle| = 2,$$

where $s_1 \in \langle s_2, s_3\rangle$, and $h \geqslant 1$, then Γ has a Hamilton decomposition.

Proof. Let $J_i := \langle s_i\rangle$, for $1 \leqslant i \leqslant 3$, so that $|A|/2 = |s_1| \geqslant |s_2| \geqslant |s_3| = m \geqslant 3$, where at least one of the inequalities is strict. It may further be assumed that $s_3 \notin J_2$, for otherwise Γ is a circulant graph, which is resolved by Theorem 1.5.2. Let $t = |A : J_2|$ and $n = |A : J_3|$. Thus $|A| = mn$, and $n = 2h_2 + 1 \geqslant t \geqslant 2$. Let $F_1 = \textsc{Cay}(A, \{s_1\})$, $F_2 = \textsc{Cay}(A, \{s_2\})$, and $F_3 = \textsc{Cay}(A, \{s_3\})$. Clearly, $\{F_1, F_2, F_3\}$ is a 2-factorization of Γ. The subgraph $F_2 \cup F_3 = \textsc{Cay}(A, \{s_2, s_3\}) \simeq A_n \,\square_r\, B_m$, where n is the number of columns, and t is the number of distinct horizontal cycles. The vertex $(i-1)s_2 + (j-1)s_3$ in $F_2 \cup F_3$ is identified with the vertex (a_i, b_j), for $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$, where $ns_2 = rs_3$, and $t = \gcd(r, m)$. Hence, $s_2 = (a_2, b_1)$ and $s_3 = (a_1, b_2)$. As $s_1 \in \langle s_2, s_3\rangle$, $s_1 = (p-1)s_2 + (q-1)s_3 \mapsto (a_p, b_q)$, for some $1 \leqslant p \leqslant n$ and $1 \leqslant q \leqslant m$. Color the edges in F_1 blue, edges in F_2 black, and edges in F_3 red.

Case 1: $\boldsymbol{n = 2h_2 + 1 > t = 2h_1 \geqslant 2}$. Here, $s_1 \notin J_3$ and $s_3 \notin J_1$ for otherwise, either $n \mid 2$ or $2 \mid n$, contradicting the choice of n. As $|A|$ is even, $m = 2s \geqslant 4$.

1. If $t = 2$ then apply an $\{a_1, a_n, b_2\}$-RAHS to $A_n \,\square_r\, B_m$ to create a red Hamilton cycle, C_R. As n is odd, the column direction pattern is: $\uparrow\downarrow\uparrow\cdots\downarrow\uparrow$. Thus, by Remark 2.2.2 and Lemma 2.2.15, apply an $\{a_n, a_1, b_1, b_2\}$-CS to obtain a black Hamilton cycle, and preserve C_R (see Figure 4.2). Give C_R an orientation so that it becomes a directed cycle. Clearly, the column direction pattern is still $\uparrow\downarrow\uparrow\cdots\downarrow\uparrow$ with the exception of the vertical $\downarrow$-edges in the a_1- column that are between the b_2- and b_{1+r}-rows. Note that $|J_1| = |J_2|$ and $A = J_1 \cup (J_1 + s_3) = J_2 \cup (J_2 + s_3)$. Additionally, $r \neq 0$, for otherwise, $ns_2 = 0 \Rightarrow |s_2| = \frac{nm}{2} \mid n \Rightarrow m \mid 2$, a contradiction. Thus, consider the cases where $J_1 = J_2$ and $J_1 \neq J_2$.
   (a) $\boldsymbol{J_1 = J_2}$: For $1 \leqslant j \leqslant m-1$ and $1 \leqslant i \leqslant n$, the vertices (a_i, b_j) and (a_i, b_{j+1}) are on different blue cycles and all elements of J_1 and J_2 are on odd b_j-rows. In particular, $s_1 = (a_p, b_q)$ for some $1 < p \leqslant n$ and $q = 2\hat{q} + 1$, where $1 \leqslant q \leqslant m-1$. We seek a vertical oblique color-switch to join the two blue cycles, and preserve C_R. Consider the following cases:
      i. $(p, q, r, m) = (2\hat{p}, 1, 2, 4)$, for some $\hat{p} \geqslant 1$. By Remark 4.1.1, $-s_1 = (a_{p^*}, b_3)$, where $p^* = n - 2\hat{p} + 2$ is odd, and $3 \leqslant p^* \leqslant n$. By Table 1.1 of the Appendix, we may assume $n \geqslant 5$. The black edges $(a_1, b_3)(a_1, b_4)$ and $(a_{2i+1}, b_1)(a_{2i+1}, b_2)$, for $i \geqslant 0$ have opposite direction. In particular, $(a_1, b_3)(a_1, b_4)$ and $(a_{p^*}, b_1)(a_{p^*}, b_2)$ have opposite direction and $(a_1, b_3)(a_{p^*}, b_1)$ and $(a_1, b_4)(a_{p^*}, b_2)$ are blue edges that are on two different cycles. Apply an $\{a_1, a_{p^*}, b_3, b_1\}$-VOCS to create a blue Hamilton cycle and, by Remark 2.2.2, break the black Hamilton cycle into a 2-factor consisting of two cycles. In particular, the edges $(a_{p^*-1}, b_2)(a_{p^*-1}, b_3)$ and $(a_{p^*}, b_2)(a_{p^*}, b_3)$ are on two different black cycles and $(a_{p^*-1}, b_2)(a_{p^*}, b_2)$ and $(a_{p^*-1}, b_3)(a_{p^*}, b_3)$ are red edges that have the same direction. Apply an $\{a_{p^*-1}, a_{p^*}, b_2, b_3\}$-CS, which by Remark 2.2.2 will create a Hamilton decomposition.
      ii. $(p, q, r, m) = (2\hat{p}, 1, 2, 2s)$, where $\hat{p} \geqslant 1$ and $s \geqslant 3$. Reflect all switches about the b_2-row. We still have a Hamilton decomposition of $A_n \,\square_r\, B_m$ but now the column direction pattern is: $\uparrow\uparrow\downarrow\cdots\uparrow\downarrow$. Hence, as $m \geqslant 6$, the edge $(a_1, b_4)(a_1, b_1)$ is red and has the same direction as $(a_p, b_m)(a_p, b_1)$. Apply an $\{a_1, a_p, b_m, b_m\}$-VOCS to obtain the result.

iii. $(p,q,r) = (2\hat{p}, 2\hat{q}+1, 2u)$, where $\hat{q} \geqslant 0$ and $u \geqslant 1$. By Figure 4.2, the vertical oblique color-switches defined in (4.1) will produce a Hamilton decomposition of Γ.

$$\text{If } (p,q,r) = (2\hat{p}, 2\hat{q}+1, 2u), \text{ apply an} \begin{cases} \{a_1, a_p, b_2, b_{q+1}\}\text{-VOCS} & \text{if } \hat{q} \geqslant 1 \\ \{a_1, a_p, b_3, b_3\}\text{-VOCS} & \text{if } \hat{q} = 0 \text{ and } u \neq 1 \end{cases} \tag{4.1}$$

iv. $p = 2\hat{p}+1$ where $\hat{p} \geqslant 1$. Cases i-iii construct Hamilton decompositions when $p = 2\hat{p}$, for some $\hat{p} \geqslant 1$. By Remark 4.1.1, $-s_1 = (a_{p^*}, b_{q^*})$, where $p^* = n - 2\hat{p} + 2$, which is even, and $q^* = m - (q + r - 2)$. Hence, apply the technique of Cases i-iii by replacing p with p^* and q with q^*.

(b) $\boldsymbol{J_1 \neq J_2}$: Here $|J_1 \cap J_2| = mn/4$, so that $x \in J_1 \cap J_2 \Leftrightarrow x = 2ks_2$, for some $k \in \mathbb{Z}$. As $s_2, s_3 \notin J_1$, (a_i, b_j) is on one blue cycle and (a_{i+1}, b_j) and (a_i, b_{j+1}) are on the other blue cycle, for all $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m-1$. All elements of J_2 occur in odd b_j-rows, so $s_1 = (a_p, b_{2\hat{q}})$ for some $1 < p \leqslant n$. If $q = 2$, then the edges $(a_1, b_1)(a_2, b_1)$ and $(a_p, b_2)(a_{p+1}, b_2)$ are red edges that have the same direction and $(a_1, b_1)(a_p, b_2)$ and $(a_2, b_1)(a_{p+1}, b_2)$ are blue edges that are not on the same cycle. By Remark 2.2.2, apply an $\{a_1, a_p, b_1, b_2\}$-HOCS to obtain a blue Hamilton cycle and preserve C_R, i.e., a Hamilton decomposition. If $p = 2\hat{p}+1$ and $\hat{q} \neq 1$, then $(a_1, b_m)(a_1, b_1)$ and $(a_p, b_{q-1})(a_p, b_q)$ are red edges that share the same direction, and we may apply an $\{a_1, a_p, b_m, b_{q-1}\}$-VOCS, which by Remark 2.2.2, preserves C_R and creates a blue Hamilton cycle. The result is a set of three monochromatic Hamilton cycles. If $p = 2\hat{p}$ and $q \neq m$, apply an $\{a_1, a_p, b_2, b_{q+1}\}$-VOCS to obtain a Hamilton decomposition. If $q = m$, then $-s_1$ lies in the odd a_{n-p+2}-column and even $b_{m-(m+r-2)}$-row. Apply a VOCS according to the case when $p = 2\hat{p}+1$ and using $-s_1$ rather than s_1 to obtain a Hamilton decomposition.

2. Suppose $t \geqslant 4$. Note, $s_1 \notin J_2$ and so $s_1 = (a_p, b_q)$ for some $1 < p \leqslant n$ and $1 < q \leqslant m$. Furthermore, the elements of J_2 consist of the vertices $(a_i, b_{1+\ell t})$, for $1 \leqslant i \leqslant n$ and $0 \leqslant \ell < k$, where $m = kt$, and therefore $q \neq 1 + \ell t$ for any $\ell \geqslant 0$. As before, the vertices (a_i, b_j) and (a_i, b_{j+1}) lie on two different blue cycles. Apply an $\{a_2, b_1, b_{t-1}\}$-LAVS and an $\{a_3, a_4, b_{t-1}, b_t\}$-CS to $F_2 \cup F_3 = A_n \,\Box_r\, B_m$ to obtain a black Hamilton cycle and join the a_i-columns, where $1 \leqslant i \leqslant 4$, into one red cycle. Apply an $\{a_4, a_{n-1}, b_t\}$-LAHS and an $\{a_{n-1}, a_n, b_t, b_{t+1}\}$-CS to create a red Hamilton cycle, C_R, and break the black Hamilton cycle into two cycles. As n is odd, the column-direction pattern is: $\uparrow\downarrow\uparrow \cdots \downarrow\uparrow$. By Remark 2.2.2 and Lemma 2.2.15, apply an $\{a_n, a_1, b_{t-1}, b_t\}$-CS to reform a black Hamilton cycle and preserve C_R. Clearly, this switch is not r-incident with any b_j-row for $1 \leqslant j \leqslant t-2$, for initially, the b_j-rows were on t different cycles, for $1 \leqslant j \leqslant t$. We now have a Hamilton decomposition of the subgraph $F_2 \cup F_3$. Make C_R a directed cycle as shown in Figure 4.3. Apply the following color-switch to obtain a Hamilton decomposition of Γ.

$$\begin{cases} \text{If } q \neq t, \text{ apply an} & \begin{cases} \{a_1, a_p, b_m, b_{q-1}\}\text{-VOCS} & \text{if } p = 2\hat{p}+1 > 3 \\ \{a_2, a_4, b_m, b_{q-1}\}\text{-VOCS} & \text{if } p = 3 \end{cases} \\ \text{If } q = t, \text{ apply an} & \begin{cases} \{a_2, a_{2\hat{p}+2}, b_{t-1}, b_{2t-1}\}\text{-VOCS} & \text{if } p = 2\hat{p}+1 < n \\ \{a_3, a_2, b_m, b_{r+t-1}\}\text{-VOCS} & \text{if } p = n \end{cases} \end{cases} \tag{4.2}$$

Each of the VOCS in (4.2) interchange the colors of two red vertical edges having the same direction, with the colors of two oblique blue edges that are on different cycles (refer to Figure 4.3). By Remark 2.2.2, the result is a Hamilton decomposition of Γ. If $p = 2\hat{p}$, then, by

Remark 4.1.1, $-s_1$ is in the odd $a_{n-2\hat{p}+2}$- column. Apply the color-switches defined in (4.2) where p is replaced with $p^* = n - p + 2$ and q is replaced with $q^* = m - (q + r - 2)$.

Case 2: $n = 2h_1 + 1 \geqslant 2h_2 + 1 = t \geqslant 3$. If $s_1 = (a_{2\hat{p}+2}, b_{2\hat{q}+1})$, for some $\hat{p}, \hat{q} \geqslant 0$, then

$$0 = \left(\tfrac{nm}{2}\right) s_1 = \tfrac{nm}{2}\left[(2\hat{p}+1)s_2 + (2\hat{q})s_3\right] = nm(\hat{p}s_2 + \hat{q}s_3) + \left(\tfrac{nm}{2}\right) s_2 = \left(\tfrac{nm}{2}\right) s_2 \Rightarrow |s_2| \mid \tfrac{nm}{2},$$

and so $\frac{nm}{t} \mid \frac{nm}{2} \Rightarrow t \mid 2$, a contradiction. Similarly, $s_1 = (a_{2\hat{p}+1}, b_{2\hat{q}+2})$ produces a contradiction. Hence, $(p, q) = (2\hat{p}, 2\hat{q})$ or $(p, q) = (2\hat{p} + 1, 2\hat{q} + 1)$. As before, $s_1 \notin J_2, J_3$, and $s_2, s_3 \notin J_1$, so $1 < p \leqslant n$, $1 < q \leqslant m$, and $q \neq 1 + \ell t$ for all $\ell \geqslant 0$. Hence, (a_i, b_j) is on one blue cycle while (a_i, b_{j+1}) and (a_{i+1}, b_j) are on the other blue cycle, for any $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$. Futhermore, $r \neq 0$, for m and t have opposite parity. If $n \geqslant 5$, apply an $\{a_2, b_1, b_t\}$-LAVS to obtain a black Hamilton cycle and join the red cycles in the a_j-columns, where $1 \leqslant j \leqslant 3$, into one red cycle. Apply an $\{a_3, a_n, b_2\}$-RAHS, which by Lemma 2.2.15, produces a Hamilton decomposition of $A_n \,\square_r\, B_m$. Give the red Hamilton cycle, C_R, an orientation so it becomes a directed cycle (refer to Figure 4.4). Similarly to Case 1(b), we apply a final switch as follows:

$$\begin{cases} \text{If } p = 2\hat{p} + 1 \neq 3, \text{ apply an} & \begin{cases} \{a_1, a_p, b_m, b_{q-1}\}\text{-VOCS} & \text{if } q \neq 3 \\ \{a_1, a_p, b_2, b_{q+1}\}\text{-VOCS} & \text{if } q = 3 \end{cases} \\ \text{If } p = 3, \text{ apply an} & \begin{cases} \{a_3, a_5, b_m, b_{q-1}\}\text{-VOCS} & \text{if } q \neq 3 \\ \{a_3, a_5, b_3, b_5\}\text{-VOCS} & \text{if } q = 3 \end{cases} \end{cases} \tag{4.3}$$

If $p = 2\hat{p}$, for some $\hat{p} \geqslant 0$, then by Remark 4.1.1, $-s_1$ is in the odd a_{n-p+2}-column, and the odd b_{q^*}-row, where $q^* = m - (q + r - 2)$. Replace p with $p^* = n - p + 2$ and q with q^* in (4.3) to obtain a Hamilton decomposition of Γ. The case $n = t = 3$, is similar, and is omitted. ■

Lemma 4.1.4. *Let $\Gamma = \mathrm{Cay}(A, \{s_1, s_2, s_3\})$ be a 6-regular, connected Cayley graph of even order, where*

$$2h = |A : \langle s_3 \rangle| \geqslant |A : \langle s_2 \rangle| \geqslant |A : \langle s_1 \rangle| = 2.$$

If $s_1 \in \langle s_2, s_3 \rangle$ and $h \geqslant 2$, then Γ has a Hamilton decomposition.

Proof. The proof is similar that of Lemma 4.1.3, where now $n = |A : J_3| = 2h \geqslant 4$ and $t = |A : J_2| \geqslant 2$. We again seek a Hamilton decomposition of $F_2 \cup F_2 = A_n \,\square_r\, B_m$ and then define HOCS or VOCS to create a blue Hamilton cycle and obtain the result.

Case 1: $n = 2h_1 > 2h_2 + 1 = t \geqslant 3$. Clearly, $s_2 \notin J_1$ and so (a_i, b_j) is on one blue cycle, while (a_{i+1}, b_j) is on the other blue cycle, for $1 \leqslant i \leqslant n$. As $s_1 \notin J_2, J_3$, we have $1 < p \leqslant n$, $1 < q \leqslant m$, and $q \neq \ell t + 1$ for any $\ell \geqslant 0$. Apply an $\{a_2, b_1, b_t\}$-LAVS to $A_n \,\square_r\, B_m$ to obtain a black Hamilton cycle and join the vertical red edges in the a_i-columns, where $1 \leqslant i \leqslant 3$, into one red cycle, C. Let $m = kt$, where $k \geqslant 1$, with equality if and only if $r = 0$. Let d be the integer, $1 \leqslant d \leqslant k$, such that $(d-1)t + 1 < q < dt + 1$. By Corollary 2.2.8, apply an $\{a_3, a_4, b_{dt}, b_{dt+1}\}$-CS to preserve the black Hamilton cycle and join the red edges in the a_4-column to C. Next, apply an $\{a_4, a_n, b_{dt}\}$-RAHS, which by Lemma 2.2.15, will preserve the black cycle and create a red Hamilton cycle. We now have a Hamilton decomposition of $A_n \,\square_r\, B_m$. Give the red Hamilton cycle an orientation, so it becomes a directed cycle (refer to Figure 4.5). We shall now define horizontal oblique color-switches to join the blue cycles and preserve the red and black Hamilton cycles. Let $j = dt - q + 1$, so that $1 \leqslant j \leqslant t - 1$. The choice of j ensures that $(a_1, b_j)(a_p, b_{dt})$ is a blue edge. To see this, note

$$(p-1)s_2 + (dt-1)s_3 - (j-1)s_3 = (p-1)s_2 + (q-1)s_3 = s_1.$$

If $d \equiv q \pmod 2$, then j is odd, and if $3 \leqslant p \leqslant n-1$, the red edges $(a_1, b_j)(a_2, b_j)$ and $(a_p, b_{dt})(a_{p+1}, b_{dt})$ have the same direction. If $d \not\equiv q \pmod 2$, then j is even, and so $(a_1, b_j)(a_2, b_j)$ and $(a_p, b_{dt})(a_{p+1}, b_{dt})$ have opposite direction. Switching on that 4-cycle will create a blue Hamilton cycle but break the red Hamilton cycle into a 2-factor with two cycles, say R_1 and R_2. The vertical red edges in the a_1-column that are between the b_j-row and the b_m-row and the edge $(a_1, b_m)(a_1, b_1)$ are on R_1, while the red edge $(a_n, b_{dt-1})(a_n, b_{dt})$ is on R_2. Furthermore, the black edges $(a_n, b_{dt-1})(a_1, b_{dt-1+r})$ and $(a_n, b_{dt})(a_1, b_{dt+r})$ have the same direction, therefore by Remark 2.2.2, the color-switches in (4.4) will produce a Hamilton decomposition of Γ. Similarly, if $p = 2$, then $q \equiv 0 \pmod 2$, and $(a_2, b_j)(a_3, b_j)$ and $(a_3, b_{dt})(a_4, b_{dt+1})$ are both red edges, and $(a_2, b_j)(a_3, b_{dt})$ and $(a_3, b_j)(a_4, b_{dt})$ are blue edges that are on different cycles, and the switches in (4.5) will yield the result.

$$\text{If } 3 \leqslant p \leqslant n-1 \qquad \begin{cases} \{a_1, a_p, b_j, b_{dt}\}\text{-HOCS} & \text{if } d \equiv q \pmod 2 \\ \{a_1, a_p, b_j, b_{dt}\}\text{-HOCS, } \{a_n, a_1, b_{dt-1}, b_{dt}\}\text{-CS} & \text{if } d \not\equiv q \pmod 2 \end{cases} \tag{4.4}$$

$$\text{If } p = 2 \qquad \begin{cases} \{a_2, a_3, b_2, b_{dt+1}\}\text{-HOCS} & \text{if } q = dt \\ \{a_2, a_3, b_j, b_{dt}\}\text{-HOCS} & \text{if } d \equiv q \pmod 2 \\ \{a_2, a_3, b_j, b_{dt}\}\text{-HOCS, } \{a_n, a_1, b_{dt-1}, b_{dt}\}\text{-CS} & \text{if } d \not\equiv q \pmod 2 \end{cases} \tag{4.5}$$

Finally, if $p = n$, then $-s_1$ is in the a_2-column. Apply the color-switches of (4.4) or (4.5), by replacing q with $q^* = m - (q + r - 2)$.

Case 2: $n = 2h_1 \geqslant 4$ and $t = 2h_2 \geqslant 2$. $m = 2s \geqslant 4$ We may assume that if $s_2 \in J_1$, then $s_3 \notin J_1$, for otherwise, A is cyclic, Γ is a circulant graph, and we are done by Theorem 1.5.2.

1. $t \geqslant 4$. We have $1 < p \leqslant n$ and $1 < q \leqslant m$.

   (a) Suppose $s_2 \in J_1$, so that $A = J_1 \cup (J_1 + s_3)$ and $(2k)s_3 \in J_1$ for all $k \geqslant 0$. Thus, x and $x + s_3$ lie on different blue cycles and as $(q-1)s_3 = s_1 - (p-1)s_2 \in J_1$, we have $q = 2\hat{q} + 1 < m$, for some $\hat{q} \geqslant 1$ and $q \notin \{\ell t, \ell t + 1\}$ for any $\ell \geqslant 0$.

      i. If $p = 2\hat{p} + 1$, then apply an $\{a_2, b_1, b_{t-1}\}$-LAVS or -RAVS, an $\{a_3, a_4, b_{t-1}, b_t\}$-CS, and an $\{a_4, a_n, b_{t-1}\}$-RAHS to $A_n \mathbin{\square_r} B_m$. By Theorem 2.2.7 and Lemma 2.2.15, the aforementioned CS-configuration produces a Hamilton decomposition of $A_n \mathbin{\square_r} B_m$. We seek a VOCS to join the two blue cycles. Give the red Hamilton cycle an orientation so it becomes a directed cycle. The column direction pattern is: $\uparrow\downarrow\uparrow\cdots\downarrow\uparrow\downarrow$. Apply the color-switches defined in (4.6) to obtain a Hamilton decomposition of Γ.

      $$\text{Apply an } \begin{cases} \{a_1, a_p, b_m, b_{q-1}\}\text{-VOCS} & \text{if } q \neq t-1 \\ \{a_1, a_p, b_2, b_{q+1}\}\text{-VOCS} & \text{if } q = t-1 \end{cases} \tag{4.6}$$

      If $p = 3$, we have the freedom to choose a LAVS or RAVS in order to guarantee that $\{(a_3, b_{q-1}), (a_3, b_q)\}$ is a red edge.

      ii. If $p = 2\hat{p}$, where $2 \leqslant p \leqslant n$, then as t is even, apply the CS-configuration defined in Lemma 2.2.9 to $A_n \mathbin{\square_r} B_m$ to obtain a black Hamilton cycle and connect the vertical red edges in the a_1 and a_n-columns into one cycle. It is easily seen that applying a $\{a_2, a_n, b_3\}$-RAHS will produce a Hamilton decomposition of $A_n \mathbin{\square_r} B_m$. Furthermore, upon orienting the red Hamilton cycle, the column-direction pattern is: $\uparrow\uparrow\downarrow\uparrow\ \cdots\downarrow\uparrow$. Apply the color-switches defined in (4.7) to obtain a Hamilton decomposition of Γ.

$$\text{Apply an } \begin{cases} \{a_1, a_p, b_m, b_{q-1}\}\text{-VOCS} & \text{if } q \neq 3 \\ \{a_1, a_p, b_1, b_q\}\text{-VOCS} & \text{if } q = 3 \text{ and } p \neq 2 \\ \{a_1, a_n, b_1, b_3\}\text{-VOCS} & \text{if } (p, q) = (2, 3) \end{cases} \tag{4.7}$$

(b) Suppose $s_2 \notin J_1$, so that $A = J_1 \cup (J_1 + s_2)$ and $(2k)s_2 \in J_1$ for all $k \geqslant 0$. Thus, x and $x + s_2$ lie on different blue cycles. We may safely assume that $p \leqslant n - 1$. If $p = n$, then $-s_1$ is in the a_2-column, by Remark 4.1.1, and we may apply the technique to $-s_1$ instead of s_1 to obtain the result.

If $m \geqslant 6$, then apply the CS-configuration of Lemma 2.2.9 and an $\{a_2, a_n, b_3\}$-RAHS to $A_n \square_r B_m$ which produces a Hamilton decomposition of $A_n \square_r B_m$ by Lemma 2.2.10. All vertical red edges in the a_1 and a_2-columns have the same direction, and all horizontal black edges in the b_j and b_{j+1}-rows have opposite directions. If $5 \leqslant q \leqslant m$, apply an $\{a_1, a_p, b_1, b_q\}$-HOCS. If $q = 2\hat{q} + 1$, we are done, for the horizontal edges have the same direction. If $q = 2\hat{q}$, a blue Hamilton cycle is created, but the black Hamilton cycle has been broken into a 2-factor consisting of two cycles. Therefore, as the b_q and b_{q+1}-rows lie on different black cycles, and the edges $(a_1, b_q)(a_1, b_{q+1})$ and $(a_2, b_q)(a_2, b_{q+1})$ are red edges with the same direction, apply an $\{a_1, a_2, b_q, b_{q+1}\}$-CS. By Remark 2.2.2, a Hamilton decomposition is obtained. The case $q \in \{2, 3, 4\}$ is similar, by using either $(a_1, b_q)(a_2, b_q)$ or $(a_1, b_{q+1})(a_2, b_{q+2})$ in place of $(a_1, b_1)(a_2, b_1)$. If $m < 6$, then $m = t = 4$ and so $r = 0$. Thus, $2 \leqslant q \leqslant 4$.

$$\begin{cases} \begin{array}{l} \{a_2, b_1, b_3\}\text{-RAVS}, \\ \{a_3, a_4, b_3, b_4\}\text{-CS}, \\ \{a_4, a_n, b_3\}\text{-RAHS} \end{array} & \text{if } q = 2 \text{ and } \begin{cases} \{a_1, a_p, b_2, b_3\}\text{-HOCS} & \text{if } p \neq 2 \\ \{a_2, a_3, b_2, b_3\}\text{-HOCS} & \text{if } p = 2 \end{cases} \\ \\ \begin{array}{l} \{a_2, b_1, b_3\}\text{-LAVS}, \\ \{a_3, a_4, b_3, b_4\}\text{-CS}, \\ \{a_4, a_n, b_3\}\text{-RAHS}, \\ \{a_1, a_p, b_1, b_3\}\text{-HOCS} \end{array} & \text{if } q = 3 \end{cases}$$

If $q = 4$, apply the CS-configuration of Lemma 2.2.9 and an $\{a_2, a_n, b_3\}$-RAHS to obtain a Hamilton decomposition of $A_n \square B_m$. Now, $(a_1, b_2)(a_2, b_2)$ and $(a_p, b_1)(a_{p+1}, b_1)$ are both black edges that have opposite direction. Furthermore, $(a_1, b_2)(a_p, b_1)$ and $(a_2, b_2)(a_{p+1}, b_1)$ are blue edges that lie on different blue cycles. Apply a $\{a_1, a_p, b_2, b_1\}$-HOCS to obtain a blue Hamilton cycle and break the black Hamilton cycle into a 2-factor where $(a_1, b_1)(a_2, b_1)$ and $(a_1, b_4)(a_2, b_4)$ are black edges on different cycles. Now, $(a_1, b_1)(a_1, b_4)$ and $(a_2, b_1)(a_2, b_4)$ are red edges with the same direction. This is because upon applying the color-switch of Lemma 2.2.10, $A_n \square B_m$ has column-direction pattern: $\uparrow\uparrow\downarrow\uparrow \cdots \downarrow\uparrow$. Apply an $\{a_1, a_2, b_4, b_1\}$-CS, which by Remark 2.2.2, yields a Hamilton decomposition.

2. If $t = 2$, then $1 < p \leqslant n$ and $1 \leqslant q \leqslant m$ and $(p, q) \neq (2, 1)$, because $s_1 \neq s_2$. Apply the CS-configuration of Lemma 2.2.9 to $A_n \square_r B_m$ to obtain a black Hamilton cycle, and join the vertical red edges in the a_1 and a_n-columns into one cycle. Consider the following cases:

(a) $s_2 \in J_1$. Here, $s_3 \notin J_1$, for otherwise, $A = J_1$, a contradiction. Hence, $J_1 = J_2$, and the vertices x and $x + s_3$ lie on different blue cycles, and the elements of J_1 consist of all vertices in the odd b_j-rows. In particular, $q = 2\hat{q} + 1 \geqslant 1$, and $1 < p \leqslant n$. Apply the following switches:

$$\begin{cases} \{a_1, a_{n-1}, b_q\}\text{-RAHS},\ \{a_1, a_p, b_1, b_q\}\text{-VOCS} & \text{if } q \neq 1, p = 2\hat{p}+1 \\ \{a_2, a_n, b_q\}\text{-RAHS},\ \{a_1, a_p, b_1, b_q\}\text{-VOCS} & \text{if } q \neq 1, p = 2\hat{p} \neq 2 \\ \{a_2, a_n, b_q\}\text{-RAHS},\ \{a_1, a_3, b_2, b_{q+1}\}\text{-VOCS} & \text{if } p = 2 \end{cases} \tag{4.8}$$

In each of the CS-configurations of (4.8), the application of the RAHS creates a red Hamilton cycle and preserves the black Hamilton cycle by Lemma 2.2.10. By orienting the black Hamilton cycle, we see that each VOCS defined in (4.8) switches blue edges on different cycles, and black edges that have the same direction. By Remark 2.2.2, a Hamilton decomposition is obtained. If $q = 1$, then $-s_1$ lies in the a_{p^*}-column and b_{q^*}-row, where $q^* = m - r + 1$. As $t = 2 \Rightarrow 0 < r < m$, we have $q^* \neq 1$. Thus, apply the CS-configuration of (4.8) by replacing p with p^* and q with q^*.

(b) $s_2 \notin J_1$. In this case, x and $x + s_2$ are on different blue cycles. Furthermore, $2 \leqslant p \leqslant n$ and $2 \leqslant q \leqslant m$, where $q = 2\hat{q}$. This is because all vertices in the odd b_j-rows are elements of J_2, and by assumption, $s_1 \notin J_2$. The result now follows exactly by using the technique of Case2.1(b). ■

Lemmata 4.1.3 and 4.1.4 combine to yield the main result of this Chapter.

Theorem 4.1.5. *If $A = \langle s_2, s_3 \rangle$ is an abelian group of even order, $|s_3| \geqslant 3$, and $|A|/2 = |s_1| \geqslant |s_2| \geqslant |s_3|$, where at least one inequality is strict, then $\mathrm{Cay}(A, \{s_1, s_2, s_3\})$ has a Hamilton decomposition.*

Corollary 4.1.6. *Let $\Delta = \mathrm{Cay}(A/\langle s_3\rangle, \{\overline{s_1}, \overline{s_2}\})$ be a quotient of $\Gamma = \mathrm{Cay}(A, \{s_1, s_2, s_3\})$ of order at least three. If $\overline{s_2}$ generates a Hamilton cycle in Δ, and $\langle s_1 \rangle$ has index 2 in A, then Γ has a Hamilton decomposition.*

Proof. As, $\overline{s_2}$ generates a Hamilton cycle, we have $\langle \overline{s_2} \rangle = A/\langle s_3 \rangle \Rightarrow A = \langle s_2, s_3 \rangle$, hence, $s_1 \in \langle s_2, s_3 \rangle$. Furthermore, $|A : \langle s_3 \rangle| \geqslant 3 > |A : \langle s_1 \rangle| = 2$, and the result follows by Theorem 4.1.5. ■

Corollary 4.1.7. *If $\Gamma = \mathrm{Cay}(\mathbb{Z}_{2m}, \{a, b, c\})$ is connected, 6-regular, $|a| = m$, and $\gcd(2m, b, c) = 1$, then Γ has a Hamilton decomposition.*

Proof. As $\mathbb{Z}_{2m}$ has exactly one subgroup of order m, namely, $A = \langle a \rangle$, and $\langle b, c \rangle = \mathbb{Z}_{2m}$, it cannot be the case that $|b| = |c| = |a|$. The result follows from Theorem 4.1.5. ■

Example 4.1.8. Let $\Gamma = \mathrm{Cay}(\mathbb{Z}_{56}, \{7, 12, 2\})$. Note $\gcd(56, 7, 12) = 1$ and $\langle 7, 12 \rangle = \mathbb{Z}_{56}$, so Γ is 6-regular and connected. Let $s_3 = 12$, $s_2 = 7$, and $s_3 = 2$. Note $|s_3| = 14$, $|s_2| = m = 8$, and $|s_3| = 28$. Recall, $\Gamma \simeq (C_7 \,\Box_4\, C_8) + F_3$, where F_3 is the 2-factor generated by 2. Here $t = \gcd(4, 8) = 4$ and $n = 7$, so $k = 2$. Apply the switching configuration shown in Figure 4.3 to the subgraph $C_7 \,\Box_4\, C_8$. As $s_3 = 2 = 6s_3 + 6s_2$, s_3 is in the a_7-column and b_7-row, i.e. $p = q = 7$. By Case 1 of Lemma 4.1.3, the application of an $\{a_2, b_1, b_3\}$-LAVS, an $\{a_3, a_4, b_3, b_4\}$-CS, an $\{a_4, a_6, b_4\}$-LAHS, an $\{a_6, a_7, b_4, b_5\}$-CS, and $\{a_7, a_1, b_3, b_4\}$-CS creates red and black Hamilton cycles. Then applying an $\{a_1, a_7, b_8, b_6\}$-VOCS gives the required Hamilton decomposition, which is illustrated in Figure 4.1.

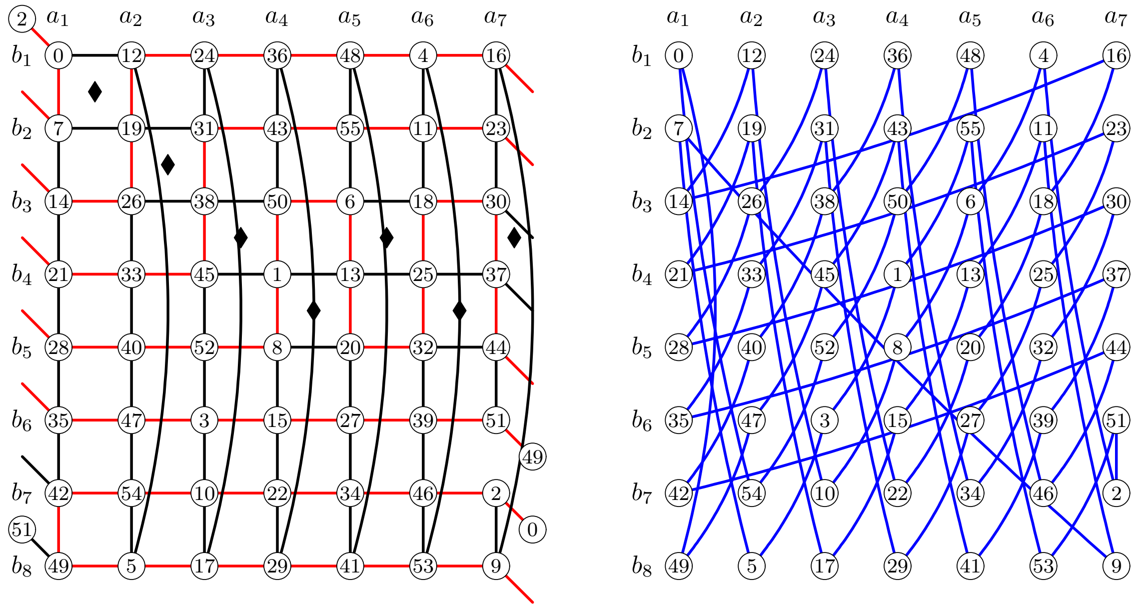


Figure 4.1: A Hamilton decomposition of $\text{Cay}(\mathbb{Z}_{56}, \{7, 12, 2\}) \simeq (A_7 \square_4 B_8) \cup \text{Cay}(\mathbb{Z}_{56}, \{2\})$, illustrating Case 2 of Lemma 4.1.3.

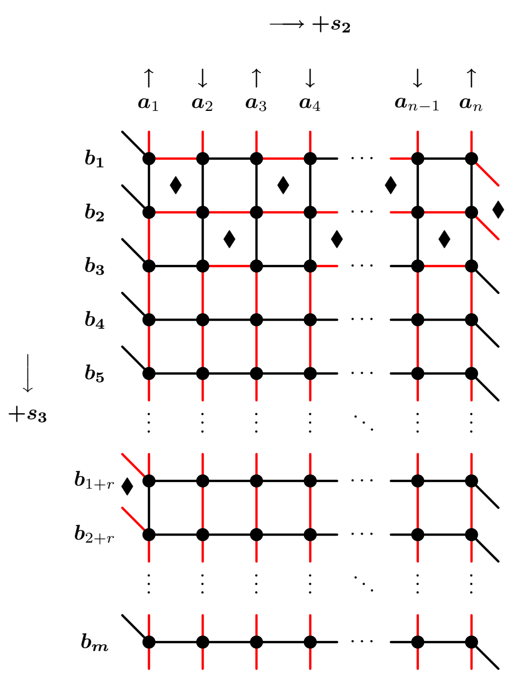


Figure 4.2: A CS-configuration of $F_2 \cup F_3$ for Case 1.1(a) of Theorem 4.1.3.

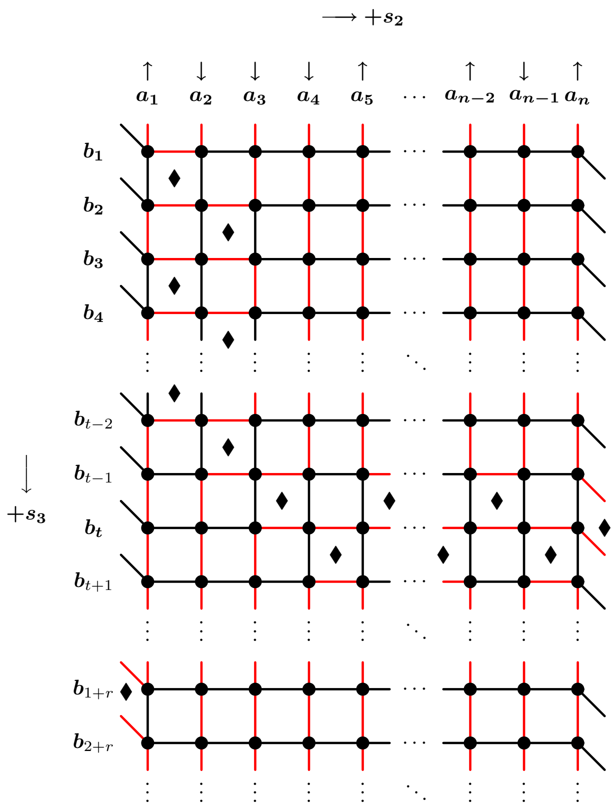


Figure 4.3: A CS-configuration of $F_2 \cup F_3$ for Case 1.2 of Theorem 4.1.3.

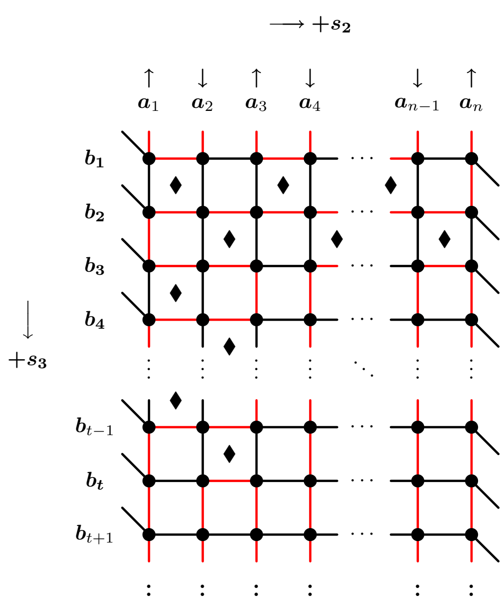


Figure 4.4: A CS-configuration of $F_2 \cup F_3$ for Case 2 of Theorem 4.1.3.

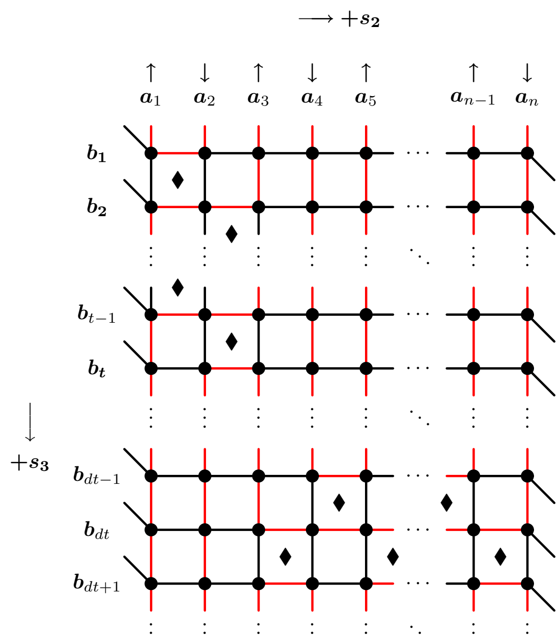


Figure 4.5: A CS-configuration of $F_2 \cup F_3$ for Case 1 of Theorem 4.1.4.

Chapter 5

Hamilton Decompositions Using Quotient Graphs

5.1 Preliminaries and Previous Work

In this chapter, the techniques used in Chapter 3 are adapted to obtain partial results for the even order, 6-regular, case of Alspach's conjecture. Recall from Remark 3.1.5, in a $D(3,m,n)$-graph, the permutations π_1 and π_2 determine the order of the columns in the two layers. We examine the graph union, $A_n^{(1)} \cup A_n^{(2)}$, whose vertices are $\{a_1, a_2, \ldots, a_n\}$, and edges $\{a_k, a_\ell\}$ if and only if either $k = \pi_1(i)$ and $\ell = \pi_1(i+1)$ for some $1 \leqslant i \leqslant n$, or $k = \pi_2(j)$ and $\ell = \pi_2(j+1)$ for some $1 \leqslant j \leqslant n$. Hence, $A_n^{(1)} \cup A_n^{(2)}$ is a 4-regular multigraph that is, by construction, Hamilton decomposable. Throughout this chapter, t_1 and t_2 will denote the number of cycles in the 2-factors, H_1 and H_2, respectively, in a $D(3,m,n)$-graph. The following lemmata were obtained in [34] and [21] by using the CS-configurations of Lemmata 2.2.9 and 2.2.11.

Lemma 5.1.1 (Liu [34]). *If $n \geqslant 8$ is even, $m \geqslant 6$, $t_1 = 2k_1 \geqslant 2$, and $\Pi_1 \cap \Pi_2 = \emptyset$, where*

$$\Pi_1 = \{\pi_1(1), \pi_1(n)\} \text{ and } \Pi_2 = \{\pi_2(i) : 1 \leqslant i \leqslant 6\},$$

for some permutations π_1 and π_2 of $[n]$, then a $D(3,m,n)$-graph has a Hamilton decomposition.

Lemma 5.1.2 (Fan et al. [21], Liu [34]). *Consider the sets*

$$\Pi_1 = \{\pi_1(1), \pi_1(2), \pi_1(3), \pi_1(4)\} \text{ and } \Pi_2 = \{\pi_2(1), \pi_2(2), \pi_2(3)\},$$

where π_1 and π_2 are permutations of $[n]$, and $n \geqslant 6$. If $\Pi_1 \cap \Pi_2 = \{\pi_1(4) = \pi_2(1)\}$, then a

$D(3,m,n)$-graph has a Hamilton decomposition if one of the following holds:

1. *$t_i = 2k_i + 1$, for $i = 1, 2$.*
2. *n is even, and $t_1 = 2k_1$ and $t_2 = 2k_2 + 1$.*

Definition 5.1.3. Suppose that G is a multigraph that can be decomposed into two Hamilton cycles C_1 and C_2. We define the following properties:

Property I : there exists a path $P = u_1u_2u_3u_4u_5$ in G such that $P_1 = u_1u_2u_3$ is on C_1 and $P_2 = u_3u_4u_5$ is on C_2.

Property II : there exists a path $P = u_1u_2u_3u_4u_5u_6$ in G such that $P_1 = u_1u_2u_3u_4$ is on C_1 and $P_2 = u_4u_5u_6$ is on C_2.

Property III : there exists a path $P = u_1u_2u_3u_4u_5u_6u_7u_8$ in G such that $P_1 = u_1u_2$ is on C_1 and $P_2 = u_3u_4u_5u_6u_7u_8$ is on C_2.

The following result follows directly from the Pigeonhole principle.

Lemma 5.1.4 (Fan et al. [21], Liu [32, 34]). *If G is a 4-regular multigraph of order n that can be decomposed into two Hamilton cycles, C_1 and C_2, then*

$$G \text{ has } \begin{cases} \textsc{Property I} & \text{if } n \geqslant 7 \\ \textsc{Property II} & \text{if } n \geqslant 9 \\ \textsc{Property III} & \text{if } n \geqslant 13 \end{cases}$$

By combining Theorems 1.5.2, 1.5.5, 3.1.9, 3.2.1, and Lemmata 5.1.1, 5.1.2, 5.1.4, we have the following summary:

Corollary 5.1.5 ([9, 15, 16, 21, 30, 32, 33, 34, 53]). *A $D(3,m,n)$-graph has a Hamilton decomposition if any one of the following are true:*

(a) $m \geqslant 3$ *is odd,* $n = 3, 5, 7$, *or* $n \geqslant 9$.

(b) $m \geqslant 4$ *is even,* $n \geqslant 9$, t_1 *and* t_2 *are odd.*

(c) $m \geqslant 4$ *is even,* $n \geqslant 10$ *is even,* t_1 *is even,* t_2 *is odd.*

(d) $m \geqslant 6$ *is even,* $n \geqslant 14$ *is even,* t_1 *and* t_2 *are even.*

Remark 5.1.6. The following cases are not covered in Corollary 5.1.5:

(e) $m \geqslant 4$ is even, $n \geqslant 9$ is odd, t_1 and t_2 are both even.

(f) $m \geqslant 4$ is even, $n \geqslant 9$ is odd, t_1 is even, t_2 is odd.

(g) $m \geqslant 4$ is even, $n = 10, 12$, t_1 and t_2 are even.

(h) $m \geqslant 3$, $n = 3, 4, 5, 6, 7, 8$.

Constructions for Cases (e)–(g) in Remark 5.1.6 are developed in Section 5.2, and constructions for Case (h) in Section 5.3.

5.2 Decomposing Layered Pseudo-Cartesian Products

Lemma 5.2.1. *If G is a multigraph with $V(G) = \{a_1, a_2, \ldots, a_n\}$, that can be decomposed into two cycles,*

$$C_1 = a_{\pi_1(1)} a_{\pi_1(2)} \cdots a_{\pi_1(n)} \text{ and } C_2 = a_{\pi_2(1)} a_{\pi_2(2)} \cdots a_{\pi_2(n)},$$

for some permutations π_1 and π_2 of $[n]$, then

1. *if $n = 12$, and G does not have* Property III*, there exists an edge u_1u_2 in C_1 dividing C_2 into two paths, each on six vertices, so w.l.o.g., $\pi_1(1) = \pi_2(6) = 1$ and $\pi_1(12) = \pi_2(12) = 12$.*
2. *if $n = 10$, there exists a path $P = u_1u_2$ in C_1 that either divides C_2 into*
   (a) *one path on six vertices and one path on four vertices, so w.l.o.g., $\pi_1(1) = \pi_2(6) = 1$, and $\pi_1(10) = \pi_2(10) = 10$, or*
   (b) *two paths on five vertices, so w.l.o.g., $\pi_1(1) = \pi_2(5)$ and $\pi_1(10) = \pi_2(10)$.*

Lemma 5.2.2. *If $t_1 = 2k_1 \geqslant 2$, $m > 2k_2 = t_2$, and $m \geqslant 6$, then a $D(3, m, 12)$-graph has a Hamilton decomposition.*

Proof. We closely follow the technique of the proof of Lemma 5.1.1 (Lemma 3.18 in [34]). If the multigraph $A_{12}^{(1)} \cup A_{12}^{(2)}$ corresponding to the $D(3, m, 12)$-graph has PIII, we are done by Lemma 5.1.1. In particular, we may assume $\pi_1 \neq \pi_2$. Thus, by Lemma 5.2.1, it is assumed that, up to a relabeling of the vertices,

$$A_{12}^{(2)} = a^2_{\pi_2(1)}\, a^2_{\pi_2(2)}\, a^2_{\pi_2(3)}\, a^2_{\pi_2(4)}\, a^2_{\pi_2(5)}\, a^2_1\, a^2_{\pi_2(7)}\, a^2_{\pi_2(8)}\, a^2_{\pi_2(9)}\, a^2_{\pi_2(10)}\, a^2_{\pi_2(11)}\, a^2_{12}\, a^2_{\pi_2(1)}.$$

Apply the CS-configuration of Lemma 2.2.9 to $H_1 \cup F \simeq A_{12} \,\square_{r_1} B_m$ to obtain a blue Hamilton cycle and join the red edges in the a_1 and a_{12}-columns into a single cycle. Note that the vertical red and blue edges in the a_1- and a_{12}-columns also form a matching. We now define color-switches in $A_{12}^{(2)} \,\square_{r_2} B_m$. Assume that $t_2 \geqslant 6$, because the proof for the cases $t_2 = 2$ or 4 is very similar, and we have more freedom to define a CS-configuration. Apply the CS-configuration of Lemma 2.2.11 to $H_2 \cup F \simeq A_{12}^{(2)} \,\square_{r_2} B_m$, by starting with $X_{1+\ell}$, where ℓ is any integer such that the edge $e = (a_1^2, b_{5+\ell})(a_1^2, b_{6+\ell})$ is red. By Remark 2.2.2 and Lemma 2.2.11, there now exists a black Hamilton cycle, and a cycle consisting of vertical red edges in the a_1^2, a_{12}^2, and $a^2_{\pi_2(i)}$-columns, where $1 \leqslant i \leqslant 5$. Call this red cycle the ★-cycle. Every column except the a_1^2 and a_{12}^2-columns, has a red path of length at least four. For any integers $2 \leqslant i \leqslant 11$ and $1 \leqslant j \leqslant m$, at least one of the two edges, $e_j = (a_i, b_j)(a_i, b_{j+1})$ and $e_{j+2} = (a_i, b_{j+2})(a_i, b_{j+3})$, is red. We now define additional color-switches in $H_1 \cup F \simeq A_{12} \,\square_{r_1} B_m$ to create three monochromatic Hamilton cycles. Let

$$\{x_1, x_2, x_3, x_4, x_5\} = \{\pi_2(i) : 7 \leqslant i \leqslant 11\}, \text{ where } 1 < x_1 < x_2 < x_3 < x_4 < x_5 < 12.$$

Define $X_i = \{a_{x_i}, a_{x_i+1}, b_{w_i}, b_{w_i+1}\}$-CS, and let $\mathcal{X} = \{X_i : 1 \leqslant i \leqslant 5\}$. By Remark 2.2.2, if $\mathcal{X}$ is applied to $A_{12} \,\square_{r_1} B_m$, the six red cycles are joined with the ★-cycle into a red Hamilton cycle, C_R. If C_R is given a direction, all vertical edges in a fixed column have the same direction, the a_{x_i} and a_{x_i+1}-columns have opposite directions, and by Lemma 2.2.9, the a_1 and a_{12}-columns have the same direction. Thus, there exists an integer x, $1 \leqslant x \leqslant 11$, such that the a_x and a_{x+1}-columns have the *same direction.* Any color-switch between these two columns will preserve the red cycle. Thus, let

$$\{x, x_1, x_2, x_3, x_4, x_5\} = \{y_i : 1 \leqslant i \leqslant 6\}, \text{ where } 1 \leqslant y_1 < y_2 < y_3 < y_4 < y_5 < y_6 < 12,$$

and define $Y_i = \{a_{y_i}, a_{y_i+1}, b_{z_i}, b_{z_i+1}\}$-CS, and let $\mathcal{Y} = \{Y_i : 1 \leqslant i \leqslant 6\}$. Upon applying the color-switches in $\mathcal{Y}$ to $A_{12} \,\Box_{r_1}\, B_m$, a red Hamilton cycle will be obtained. We now define good pairs (see Definition 2.2.14) $\{Y_1, Y_2\}$, $\{Y_3, Y_4\}$, and $\{Y_5, Y_6\}$, so that applying $\mathcal{Y}$ will preserve the blue Hamilton cycle, and yield the result. There are two cases:

1. $x \notin \{y_{2j-1}, y_{2j}\}$. The $a_{y_{2j-1}}$-column has at most one non-red edge, the $a_{y_{2j}}$-column has all red edges, and $1 < y_{2j-1} < 11$. Let $y \equiv 0 \pmod 2$ be any integer such that there exists a red 3-path in the $a_{y_{2j-1}}$-column between the b_y and b_{y+3}-rows. Let

$$e_y = (a_{y_{2j-1}+1}, b_y)(a_{y_{2j-1}+1}, b_{y+1}) \quad \text{and} \quad f_y = (a_{y_{2j}+1}, b_y)(a_{y_{2j}+1}, b_{y+1}).$$

   At least one of e_y and e_{y+2} is red. Define the good pair $\{Y_{2j-1}, Y_{2j}\}$ as follows:

$$Y_{2j-1} = \begin{cases} \{a_{y_{2j-1}}, a_{y_{2j-1}+1}, b_y, b_{y+1}\}\text{-CS} & \text{if } e_y \text{ is red} \\ \{a_{y_{2j-1}}, a_{y_{2j-1}+1}, b_{y+2}, b_{y+3}\}\text{-CS} & \text{if } e_y \text{ is not red} \end{cases} \tag{5.1}$$

$$Y_{2j} = \begin{cases} \{a_{y_{2j}}, a_{y_{2j}+1}, b_{y+1}, b_{y+2}\}\text{-CS} & \text{if } f_{y+1} \text{ is red} \\ \{a_{y_{2j}}, a_{y_{2j}+1}, b_{y-1}, b_y\}\text{-CS} & \text{if } f_{y+1} \text{ is not red, } e_y \text{ is red} \\ \{a_{y_{2j}}, a_{y_{2j}+1}, b_{y+3}, b_{y+4}\}\text{-CS} & \text{if } f_{y+1} \text{ is not red, } e_y \text{ is not red} \end{cases} \tag{5.2}$$

2. $x \in \{y_{2j-1}, y_{2j}\}$. Clearly, $x \in \{1, \pi_2(1), \ldots, \pi_2(5)\}$, the a_x-column has at most one blue edge, and the a_{x+1}-column has no blue edges. Furthermore, by Equation (5.2), any blue edge in the a_x-column lies between the b_z and b_{z+1}-rows, for some integer $z \equiv 1 \pmod 2$.

   (a) $x = y_{2j-1}$. If $x = 1$, select an integer $y \equiv 0 \pmod 2$ such that both

$$(a_1, b_y)(a_1, b_{y+1}) \text{ and } (a_1, b_{y+2})(a_1, b_{y+3})$$

   are red. Define Y_{2j-1} as in Equation (5.1). If $x \neq 1$, let P be a longest path of red edges in the a_{x+1}-column. Because $x + 1 \neq 1, 12$, and $m \geqslant 2t_2$, P has length at least four. Therefore, there exists an integer $y \equiv 0 \pmod 2$ such that both $(a_{x+1}, b_y)(a_{x+1}, b_{y+1})$ and $(a_{x+1}, b_{y+2})(a_{x+1}, b_{y+3})$ lie on P. If $e_y = (a_x, b_y)(a_x, b_{y+1})$ is red, then define $Y_{2j-1} = \{a_x, a_{x+1}, b_y, b_{y+1}\}$-CS. If e_y is black, then define $Y_{2j-1} = \{a_x, a_{x+1}, b_{y+2}, b_{y+3}\}$-CS. Having defined Y_{2j-1}, it is clear that the $a_{y_{2j}}$-column contains at most one non-red edge. Define Y_{2j} similarly to Equation (5.2) to obtain a good pair $\{Y_{2j-1}, Y_{2j}\}$.

   (b) $x = y_{2j}$. There exists an integer $y \equiv 1 \pmod 2$, such that both $(a_{y_{2j}}, b_y)(a_{y_{2j}}, b_{y+1})$ and $(a_{y_{2j}+1}, b_y)(a_{y_{2j}+1}, b_{y+1})$ are red. Define $Y_{2j} = \{a_{y_{2j}}, a_{y_{2j}+1}, b_y, b_{y+1}\}$-CS. Again, the $a_{y_{2j-1}}$-column contains all red edges, except possibly one blue edge

$$(a_{y_{2j-1}}, b_z)(a_{y_{2j-1}}, b_{z+1}),$$

   for some $z \equiv 1 \pmod 2$. Thus, either

$$Y_{2j-1} = \{a_{y_{2j-1}}, a_{y_{2j-1}+1}, b_{y-1}, b_y\}\text{-CS}$$

   or

$$Y_{2j-1} = \{a_{y_{2j-1}}, a_{y_{2j-1}+1}, b_{y+1}, b_{y+2}\}\text{-CS}$$

   is a good switch, making $\{Y_{2j-1}, Y_{2j}\}$ a good pair.

Finally, the blue Hamilton cycle is preserved upon applying Y_{2j-1} and Y_{2j}. To see this, after applying

Y_{2j-1}, the blue Hamilton cycle is broken into one cycle

$$(a_{y_{2j-1}+1}, b_{\hat{y}})(a_{y_{2j-1}+1}, b_{\hat{y}+1})(a_{y_{2j-1}+2}, b_{\hat{y}+1}) \cdots (a_{12}, b_{\hat{y}+1})(a_{12}, b_{\hat{y}})(a_{11}, b_{\hat{y}}) \cdots (a_{y_{2j-1}+1}, b_{\hat{y}}),$$

for some $\hat{y} \equiv 0 \pmod 2$, and one cycle on the remaining vertices. By Remark 2.2.2, Y_{2j} restores the blue Hamilton cycle. ■

Lemma 5.2.3. *If $t_1 = 2k_1 \geqslant 2$ and $t_2 = m = 2k \geqslant 6$, then a $D(3, m, 12)$-graph has a Hamilton decomposition.*

Proof. We proceed similarly to the proof of Lemma 5.2.2 with only slight modifications. Again, apply the CS-configuration of Lemma 2.2.9 to $H_1 \cup F \simeq A_{12} \,\square_{r_1}\, B_m$.

1. $m \equiv 2 \pmod 4$. Apply the CS-configuration of Lemma 2.2.11 to $H_2 \cup F \simeq A_{12}^{(2)} \,\square_{r_2}\, B_m$, by starting with $X_{1+\ell}$, where ℓ is any integer such that the edge

$$e = (a_1^2, b_{5+\ell})(a_1^2, b_{6+\ell})$$

is red. For any integers $2 \leqslant i \leqslant 11$ and $1 \leqslant j \leqslant m$, at least one of the two edges, $e_j = (a_i, b_j)(a_i, b_{j+1})$ and $e_{j+2} = (a_i, b_{j+2})(a_i, b_{j+3})$, is red. In this case, every column other than the a_1^2 and a_{12}^2-columns contain either at least a 4-path of red edges, or contain an 8-path, consisting of three red edges, then two black edges, then three red edges again. We denote the former P_4^R and the latter $P_3^R - P_2^B - P_3^R$. We construct the CS-configuration

$$\mathcal{Y} = \{Y_i : 1 \leqslant i \leqslant 6\},$$

defined in Lemma 5.2.2 to again preserve the blue Hamilton cycle. If $x \notin \{y_{2j-1}, y_{2j}\}$, define the good pair $\{Y_{2j-1}, Y_{2j}\}$ as in Equations (5.1) and (5.2). If $x \in \{y_{2j-1}, y_{2j}\}$, then the a_x-column initially had either a P_4 or $P_3^R - P_2^B - P_3^R$. Again, any blue edge lies between the b_z and b_{z+1}-rows, for some integer $z \equiv 1 \pmod 2$. It is easy to see that there exists an integer $y \equiv 0 \pmod 2$ such that both $(a_{x+1}, b_y)(a_{x+1}, b_{y+1})$ and $(a_{x+1}, b_{y+2})(a_{x+1}, b_{y+3})$ are red. Apply the technique of Case 2 of Lemma 5.2.2 to define a good switch.

2. $m \equiv 0 \pmod 4$. Let $m = 8 + 4d$, where $d \geqslant 0$. Define

$$D_i = \{a^2_{\pi_2(i)}, a^2_{\pi_2(i+1)}, b_{i+\ell}, b_{i+1+\ell}\}\text{-CS},$$

and apply the CS-configuration $\mathcal{D} = \{D_i : 1 \leqslant i \leqslant 5\}$, where ℓ is chosen so that e, defined as in Case 1, is red. Apply an $\{a_1^2, a^2_{\pi_2(8)}, b_{7+\ell}\}$-LAHS. If $m > 8$, define the CS-configuration,

$$\mathcal{E}_i = \begin{cases} \{a^2_{\pi_2(1)}, a^2_{\pi_2(2)}, b_{8+4i+\ell}, b_{8+4i+1+\ell}\}\text{-CS}, \\ \{a^2_{\pi_2(2)}, a^2_{\pi_2(3)}, b_{9+4i+\ell}, b_{9+4i+1+\ell}\}\text{-CS}, \\ \{a^2_{\pi_2(3)}, a^2_{\pi_2(4)}, b_{10+4i+\ell}, b_{10+4i+1+\ell}\}\text{-CS}, \\ \{a^2_{\pi_2(4)}, a^2_{\pi_2(5)}, b_{11+4i+\ell}, b_{11+4i+1+\ell}\}\text{-CS} \end{cases}$$

Apply $\mathscr{E} = \{\mathcal{E}_i : 0 \leqslant i \leqslant d-1\}$. We now have a black Hamilton cycle, and a red 2-factor consisting of one cycle on the a_{12}^2 and $a^2_{\pi_2(i)}$-columns, where $1 \leqslant i \leqslant 8$, and three cycles, one on each of the $a^2_{\pi_2(i)}$-columns, where $9 \leqslant i \leqslant 11$. Furthermore, every column, except the a_1^2 and a_{12}^2-columns, have a red 5-path, and for any integers $2 \leqslant i \leqslant 11$ and $1 \leqslant j \leqslant m$, at least one of the two edges, $e_j = (a_i, b_j)(a_i, b_{j+1})$ and $e_{j+2} = (a_i, b_{j+2})(a_i, b_{j+3})$, is red. Use the method of Lemma 5.2.2 to obtain the result. ∎

Lemma 5.2.4. *Let* $\Gamma = \textsc{Cay}(A, \{s_1, s_2, s_3\})$ *be a connected, 6-regular abelian Cayley graph. If* $|s_3| \geqslant 6$*,* $[A : \langle s_3\rangle] = 10$*, and* $2s_1, 2s_2 \notin \langle s_3\rangle$*, then* Γ *has a Hamilton decomposition.*

Proof. Γ is a $D(3, m, 10)$-graph, and by Corollary 5.1.5, we may assume t_1 and t_2 are even. $A/\langle s_3\rangle \cong \mathbb{Z}_{10}$, and so

$$\Delta = \textsc{Cay}(A/\langle s_3\rangle, \{\overline{s_1}, \overline{s_2}\}) \simeq \textsc{Cay}(\mathbb{Z}_{10}, \{x, y\}),$$

where $\langle x, y\rangle = \mathbb{Z}_{10}$, and $|x|, |y| \geqslant 3$. Without loss of generality, up to a relabeling of the vertices,

$$\{x, y\} \in \{\{\pm 1\}, \{\pm 3\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{3, 4\}\}.$$

If $\{x, y\} \in \{\{\pm 1\}, \{\pm 3\}\}$, then Δ is a multigraph. Thus $\overline{H_1} = \overline{H_2}$, and both cycles are generated by a single element. Trivially, the path $P = 0, 1, 2, 3, 4, 5, 6, 7$ has Property III of Lemma 5.1.4, so Γ has a Hamilton decomposition by Lemma 5.1.1. If $\{x, y\} = \{1, 3\}$, then

$$\overline{H_1} := 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 0 \quad \text{and} \quad \overline{H_2} := 0, 3, 6, 9, 2, 5, 8, 1, 4, 7, 0$$

is a Hamilton decomposition, and $P = 0, 3, 4, 5, 6, 7, 8, 9$ has Property III. Similarly, if $\{x, y\} = \{1, 2\}$, then

$$\overline{H_1} := 0, 2, 1, 3, 4, 5, 6, 7, 8, 9, 0 \quad \text{and} \quad \overline{H_2} := 7, 5, 3, 2, 4, 6, 8, 0, 1, 9, 7$$

is a Hamilton decomposition, and $P = 0, 9, 7, 5, 3, 2, 4, 6$ has Property III. If $\{x, y\} = \{2, 3\}$, then

$$\overline{H_1} := 0, 2, 9, 1, 3, 5, 8, 6, 4, 7, 0 \quad \text{and} \quad \overline{H_2} := 5, 2, 4, 1, 8, 0, 3, 6, 9, 7, 5$$

is a Hamilton decomposition of Δ. Now, by Lemma 5.2.1, we may assume $\pi_1(1) = \pi_2(6) = 1$ and $\pi_1(10) = \pi_2(10) = 10$. Thus, we may use the technique of Lemma 5.2.2 to find a Hamilton decomposition of Γ. The result is obtained similarly for $\{x, y\} = \{1, 4\}$ and $\{x, y\} = \{3, 4\}$. ∎

Lemma 5.2.5. *If* $\Delta = \textsc{Cay}(B, \{s_1, s_2\})$ *is a 4-regular, abelian Cayley graph on* $B = \{b_1, b_2, \ldots, b_n\}$*, and* $E(\Delta) = H_1 \cup H_2$ *is a Hamilton decomposition of* Δ*, where*

$$H_k := b_{\pi_k(1)} b_{\pi_k(2)} b_{\pi_k(3)} \cdots b_{\pi_k(n)} b_{\pi_k(1)},$$

for $k = 1, 2$*, such that* $\pi_1(i) = \pi_2(j)$ *for some* $1 \leqslant i, j \leqslant n$*, then*

$$|\{\pi_2(j-1), \pi_2(j+1)\} \cap \{\pi_1(i-1), \pi_1(i+1)\}| \geqslant 1 \Leftrightarrow s_1 = \pm s_2.$$

Proof. If $s_1 = \pm s_2$, then Δ is a multigraph, and so $H_1 = H_2$, and the result follows. Conversely, if

$$\pi_1(i+1) \in \{\pi_2(j-1), \pi_2(j+1)\}$$

or

$$\pi_1(i-1) \in \{\pi_2(j-1), \pi_2(j+1)\},$$

then either the edge $e = \{b_{\pi_1(i-1)}, b_{\pi_1(i)}\}$ or $f = \{b_{\pi_1(i)}, b_{\pi_1(i+1)}\}$ appears on both H_1 and H_2. Without loss of generality, suppose e is a multiedge, and $b_{\pi_1(i-1)} - b_{\pi_1(i)} = s_1$. If $b_{\pi_1(i-1)} - b_{\pi_1(i)} = -s_1$, then $|s_1| = 2$, a contradiction, as $|s_i| \geqslant 3$. Thus, we must have $s_1 = \pm s_2$, so that $H_1 = H_2$. ∎

Remark 5.2.6. By Lemma 5.2.5, any $D(3,m,n)$-graph arising from a 6-regular Cayley graph $\textsc{Cay}(A,\{s_1,s_2,s_3\})$ with a 4-regular quotient $\textsc{Cay}(A/\langle s_3\rangle,\{\overline{s_1},\overline{s_2}\})$, will have two columns adjacent in both $H_1\cup F=A_n^{(1)}\,\Box_{r_1}\,B_m$ and $H_2\cup F=A_n^{(2)}\,\Box_{r_2}\,B_m$ if and only if $\overline{s_1}=\pm\overline{s_2}$.

Lemma 5.2.7. *If* $\Gamma=\textsc{Cay}(A,\{s_1,s_2,s_3\})$ *is a connected, 6-regular, Cayley graph on A, where* $|s_3|\geqslant 5$, $[A:\langle s_3\rangle]\geqslant 9$, *and* $2s_1,2s_2\notin\langle s_3\rangle$, *then* Γ *has a Hamilton decomposition.*

Proof. We follow the technique and notation of ([32], Lemma 3.14). Let $J=\langle s_3\rangle$, $|J|=m$, and $[A:J]=n$. Clearly, $\Delta=\textsc{Cay}(A/J,\{\overline{s_1},\overline{s_2}\})$ is 4-regular and connected, where $A/J=\{\overline{a_1},\overline{a_2},\dots,\overline{a_n}\}$. By Theorem 1.5.1, Δ has a Hamilton decomposition into two cycles:

$$\overline{H_1}=\overline{a_{\pi_1(1)}}\,\overline{a_{\pi_1(2)}}\cdots\overline{a_{\pi_1(n)}}\,\overline{a_{\pi_1(1)}}\text{ and }\overline{H_2}=\overline{a_{\pi_2(1)}}\,\overline{a_{\pi_2(2)}}\cdots\overline{a_{\pi_2(n)}}\,\overline{a_{\pi_2(1)}},$$

for some permutations π_1 and π_2 of $[n]$. By Theorem 3.1.6, Γ is a $D(3,m,n)$-graph and by Remark 3.1.5, $H_1\cup F\simeq A_n^{(1)}\,\Box_{r_1}\,B_m$ and $H_2\cup F\simeq A_n^{(2)}\,\Box_{r_2}\,B_m$, where F is the 2-factor generated by s_3, and the 2-factors H_1 and H_2 each consist of t_1 and t_2 cycles, respectively. By Theorem 5.1.5, we may assume $m=2k\geqslant 6$ and $n=2x+1\geqslant 9$. Hence, there are exactly two cases to consider: (1) both t_1 and t_2 are even, and (2) one of t_1 and t_2 are even, the other is odd.

Case 1: Let $t_i=2k_i\geqslant 2$ for $i=1,2$.

I. $\overline{s_1}=\pm\overline{s_2}$. Without loss of generality, $\pi_1=\pi_2=(1)$. Apply the CS-configuration of Lemma 2.2.11 to $A_n\,\Box_{r_1}\,B_m$ to obtain a blue Hamilton cycle and join the vertical red cycles in the a_i-columns into one red cycle for $1\leqslant i\leqslant d$, where $d=2$ if $t_1=2$, $d=4$ if $t_1=4$, and $d=6$ if $t_1\geqslant 6$. For convenience of notation, we may assume $t_1\geqslant t_2$.

   (a) If $t_1\geqslant 6$, let h be the integer such that $e=(a_6^2,b_{1+h})(a_6^2,b_{2+h})$ is blue. If $t_2\geqslant 4$, then apply an $\{a_7^2,b_{1+h},b_{t_2-1+h}\}$-RAVS to $H_2\cup F=A_n^{(2)}\,\Box_{r_2}\,B_m$. By Theorem 2.2.7(b), this creates a black Hamilton cycle and joins the vertical red edges in the a_i-columns where $1\leqslant i\leqslant 9$ into one cycle. Let $d=n-9$. As $d\equiv 0\pmod 2$, we may apply $\{a_9^2,a_n^2,b_{t_2-1+h}\}$-RAHS to $H_2\cup F$ to join the red edges in the a_i-columns with $1\leqslant i\leqslant n$ into a Hamilton cycle and preserve the black cycle, by Lemma 2.2.15. The result is a Hamilton decomposition of Γ. If $t_2=2$, apply an $\{a_6^2,a_7^2,b_{2+h},b_{3+h}\}$-CS to $H_2\cup F$ to obtain a black Hamilton cycle and join all red edges in the a_i-columns with $1\leqslant i\leqslant 7$ into one red cycle. Then apply an $\{a_7^2,a_n^2,b_{2+h}\}$-RAHS to $H_2\cup F$ to join the red edges in the a_i-columns with $1\leqslant i\leqslant n$ into a Hamilton cycle, preserve the black cycle, and obtain a Hamilton decomposition of Γ.

   (b) If $t_1\in\{2,4\}$, apply to $H_1\cup F$ an $\{a_2,a_6,b_2\}$-LAHS if $t_1=2$ or an $\{a_4,a_6,b_3\}$-RAHS if $t_1=4$. This joins all vertical red edges in the a_i-columns with $1\leqslant i\leqslant 6$ into one cycle, and preserves the blue Hamilton cycle by Lemma 2.2.15. Apply the technique of (a) to $H_2\cup F$ to obtain the result.

II. $\overline{s_1}\neq\pm\overline{s_2}$. By Lemma 5.1.4, Δ has Property II, so there exists a path P in Δ, where

$$P=\overline{a_{\pi_1(1)}}\,\overline{a_{\pi_1(2)}}\,\overline{a_{\pi_1(3)}}\,\overline{a_{\pi_1(4)}}\,\overline{a_{\pi_2(2)}}\,\overline{a_{\pi_2(3)}}.$$

W.l.o.g., we may take π_1 to be a cyclic permutation of $[n]$, so that $\pi_1(4)=\pi_2(1)=p$, and $\{\pi_2(2),\pi_2(3)\}=\{q,n\}$, where $4\leqslant p<q<n$. Thus, relocate the r_1-jump to be between the a_1 and a_n-columns rather than the $a_{\pi_1(1)}$ and $a_{\pi_1(n)}$-columns. Thus,

$$A_n^{(1)}=a_1a_2a_3\cdots a_p\cdots a_q\cdots a_{n-1}a_na_1,\text{ and }A_n^{(2)}=a_p^2a_{\pi_2(2)}^2a_{\pi_2(3)}^2\cdots a_{\pi_2(n)}^2a_p^2,$$

where, by Remark 5.2.6, $q \neq n-1$. This time, as opposed to Case 1.I, assume $t_2 \geqslant t_1$. Apply the CS-configuration to $H_1 \cup F \simeq A_n^{(1)} \square_{r_1} B_m$,

$$\begin{cases} \{\{a_1, a_2, b_2, b_3\}\text{-CS}, \{a_2, a_3, b_1, b_2\}\text{-CS}, \{a_3, a_4, b_2, b_3\}\text{-CS}\} & \text{if } t_1 = 2 \\ \{\{a_2, b_1, b_{t_1-1}\}\text{-RAVS}, \{a_3, a_4, b_{t_1-1}, b_{t_1}\}\text{-CS}\} & \text{if } t_1 \geqslant 4 \end{cases} \tag{5.3}$$

Upon applying (5.3), by Remark 2.2.2, and either Lemma 2.2.15 or Theorem 2.2.7(b), a blue Hamilton cycle is created, and red edges in the a_i-columns are joined into a single cycle, where $1 \leqslant i \leqslant 4$. Next, as n is odd, apply the following CS-configuration to $H_1 \cup F$:

$$\begin{cases} \{a_4, a_{n-1}, b_1\}\text{-RAHS or an } \{a_4, a_{n-1}, b_2\}\text{-LAHS} & \text{if } t_1 = 2 \\ \{a_4, a_{n-1}, b_{t_1-1}\}\text{-RAHS or an } \{a_4, a_{n-1}, b_{t_1}\}\text{-LAHS} & \text{if } t_1 \geqslant 4 \end{cases} \tag{5.4}$$

By Lemma 2.2.15, the color-switches in (5.4) will preserve the blue Hamilton cycle. Remove the two color-switches that are incident to the a_q-column. The blue Hamilton cycle is still preserved, by Corollary 2.2.16, and there exists a red 2-factor consisting of four cycles, call them the $\diamondsuit$, $\clubsuit$, $\heartsuit$, and $\spadesuit$-cycles*, where

(a) the $\diamondsuit$-cycle consisting of the a_i-columns with $1 \leqslant i \leqslant q-1$,

(b) the $\clubsuit$-cycle of red edges of the $a_{\pi_2(2)}$-column,

(c) the $\heartsuit$-cycle of red edges of the $a_{\pi_2(3)}$-column, and

(d) the $\spadesuit$-cycle consisting of the a_i-columns with $q+1 \leqslant i \leqslant n-1$.

Note, the a_p-column contains exactly one blue edge if $p = q-1$, (which, by Lemma 5.2.5, is only possible if $q = \pi_2(3)$) or two consecutive blue edges if $p \neq q-1$. In the latter case, these edges are $\{e_1, e_2\}$ if a LAHS was applied in (5.4) or $\{e_1, e_3\}$ if a RAHS was applied in (5.4):

$$\begin{aligned} e_1 &= (a_p, b_y)(a_p, b_{y+1}), \\ e_2 &= (a_p, b_{y+1})(a_p, b_{y+2}), \\ e_3 &= (a_p, b_{y-1})(a_p, b_y), \end{aligned}$$

and $y = 1$ if $t_1 = 2$ or $y = t_1 - 1$ if $t_1 \geqslant 4$. We now define a CS-configuration for $H_2 \cup F$. Let ℓ be the integer such that $(a_p^2, b_{m-1+\ell}) = (a_p, b_y)$. In $H_2 \cup F \simeq A_n^{(2)} \square_{r_2} B_m$, the edges e_1, e_2, and e_3 are:

$$\begin{aligned} e_1 &= (a_p^2, b_{m-1+\ell})(a_p^2, b_{m+\ell}) \\ e_2 &= (a_p^2, b_{m+\ell})(a_p^2, b_{1+\ell}) \\ e_3 &= (a_p^2, b_{m-2+\ell})(a_p^2, b_{m-1+\ell}) \end{aligned}$$

Apply the following CS-configuration to $H_2 \cup F \simeq A_n^{(2)} \square_{r_2} B_m$:

$$\begin{cases} \{a^2_{\pi_2(2)}, b_{1+\ell}, b_{3+\ell}\}\text{-LAVS or -RAVS} & \text{if } t_2 = 2 \\ \{a^2_{\pi_2(2)}, b_{1+\ell}, b_{t_2-1+\ell}\}\text{-LAVS or -RAVS} & \text{if } t_2 \geqslant 4 \end{cases} \tag{5.5}$$

(Note, if $t_2 = m$ and e_3 is blue, we are forced to apply a LAVS in (5.5).) By Remark 2.2.2, the color-switches of (5.5) create a black 2-factor consisting of two cycles. Furthermore, as these

*named in honor of Brian Alspach, who is an avid poker player and poker author.

color-switches are between the a^2_p, a^2_q, and a^2_n-columns (in some order), a red 2-factor is created that consists of the ♠-cycle, and a new cycle formed by joining the ♢, ♣, and ♡-cycles together, which will be called the ★-cycle. Let $3 \leqslant z \leqslant n-1$ be the smallest integer such that the red edges in the $a^2_{\pi_2(z)}$-column are in the ★-cycle, but the red edges in the $a^2_{\pi_2(z+1)}$-column are in the ♠-cycle. By Remark 2.2.2, a color-switch, X, between the $a^2_{\pi_2(z)}$ and $a^2_{\pi_2(z+1)}$-columns will create a red Hamilton cycle. The ♠-cycle $a^2_{\pi_2(i)}$-columns have either two consecutive blue edges, or exactly one blue edge if $\pi_2(i) \in \{q+1, n-1\}$. On the other hand, the ★-cycle $a^2_{\pi_2(i)}$-columns, where $4 \leqslant i \leqslant n-1$, have the following possible forms:

i. $\pi_2(i) = 1$, thus containing an alternating path of $(t_1-2)/2$ red and blue edges when $t_1 \geqslant 4$ or one blue edge when $t_1 = 2$.

ii. $\pi_2(i) = 2$, thus containing a path of $t_1 - 2$ blue edges when $t_1 \geqslant 4$ or two consecutive blue edges when $t_1 = 2$.

iii. $\pi_2(i) = 3$, thus containing an alternating path of $t_1/2$ red and blue edges when $t_1 \geqslant 4$ or two consecutive blue edges when $t_1 = 2$.

iv. contain either one or two (consecutive) blue edges. Call these columns, $\bigstar_2$-columns.

We will now define X according to which property (i)–(iv), the $a^2_{\pi_2(z)}$-column has.

(a) The $a^2_{\pi_2(z)}$-column is a $\bigstar_2$-column. First consider $m > t_2$ and $z \neq 3$. If $t_2 \geqslant 4$, then consider the edges g_i and h_i:

$$g_i := (a^2_{\pi_2(z)}, b_{t_2-1+\ell+i})(a^2_{\pi_2(z)}, b_{t_2+\ell+i})$$

$$h_i := (a^2_{\pi_2(z+1)}, b_{t_2-1+\ell+i})(a^2_{\pi_2(z+1)}, b_{t_2+\ell+i})$$

If both g_0 and h_0 are red, define

$$X := \{a^2_{\pi_2(z)}, a^2_{\pi_2(z+1)}, b_{t_2-1+\ell}, b_{t_2+\ell}\}\text{-CS},$$

to obtain a Hamilton decomposition, by Remark 2.2.2. If both g_0 and h_0 are blue, then because $m \geqslant 2t_2$, both $(a^2_{\pi_2(z)}, b_{m+\ell})(a^2_{\pi_2(z)}, b_{1+\ell})$ and $(a^2_{\pi_2(z+1)}, b_{m+\ell})(a^2_{\pi_2(z+1)}, b_{1+\ell})$ are red. Furthermore, the black edges

$$(a_{\pi_2(z)}, b_{m+\ell})(a^2_{\pi_2(z+1)}, b_{m+\ell}) \text{ and } (a_{\pi_2(z)}, b_{1+\ell})(a^2_{\pi_2(z+1)}, b_{1+\ell})$$

lie on different cycles. Thus, define

$$X := \{a^2_{\pi_2(z)}, a^2_{\pi_2(z+1)}, b_{m+\ell}, b_{1+\ell}\}\text{-CS},$$

which by Remark 2.2.2 will produce a Hamilton decomposition. If g_0 is blue and h_0 is red, then clearly, both g_2 and $(a^2_{\pi_2(z)}, b_{m+\ell})(a^2_{\pi_2(z)}, b_{1+\ell})$ are red. Futhermore, either h_2 is red or $w = (a^2_{\pi_2(z+1)}, b_{m+\ell})(a^2_{\pi_2(z+1)}, b_{1+\ell})$ is red. If w is red, define

$$X := \{a^2_{\pi_2(z)}, a^2_{\pi_2(z+1)}, b_{m+\ell}, b_{1+\ell}\}\text{-CS},$$

or if h_2 is red, define

$$X := \{a^2_{\pi_2(z)}, a^2_{\pi_2(z+1)}, b_{t_2+1+\ell}, b_{t_2+2+\ell}\}\text{-CS},$$

and replace the RAVS or LAVS of (5.5) with an $\{a^2_{\pi_2(2)}, b_{3+\ell}, b_{t_2+1+\ell}\}$-LAVS or RAVS. The result, by Remark 2.2.2, is a Hamilton decomposition. If g_0 is red and h_0 is blue, use the same technique to find a good switch. In the case $t_2 = 2$, and

$$(a^2_{\pi_2(z)}, b_{1+\ell})(a^2_{\pi_2(z)}, b_{2+\ell}) \text{ and } (a^2_{\pi_2(z+1)}, b_{1+\ell})(a^2_{\pi_2(z+1)}, b_{2+\ell})$$

are red edges, then define

$$X := \{a^2_{\pi_2(z)}, a^2_{\pi_2(z+1)}, b_{1+\ell}, b_{2+\ell}\}\text{-CS}$$

to obtain a Hamilton decomposition. If not, as $m \geqslant 6$, apply a similar technique as when $t_2 \geqslant 4$ to obtain the result.

Now consider $m = t_2$, i.e., $H_2 \cup F \simeq A^{(2)}_n \,\square\, B_m$. W.l.o.g., assume that e_1 and e_2 are blue edges, and define g_i and h_i as before.

$$X = \begin{cases} \{a^2_{\pi_2(z)}, a^2_{\pi_2(z+1)}, b_{m-1+\ell}, b_{m+\ell}\}\text{-CS} & \text{if } g_0 \text{ and } h_0 \text{ are both red} \\ \{a^2_{\pi_2(z)}, a^2_{\pi_2(z+1)}, b_{m+\ell}, b_{1+\ell}\}\text{-CS} & \text{if } g_1 \text{ and } h_1 \text{ are both red} \end{cases} \tag{5.6}$$

If g_{-1} and h_{-1} are both red, define $X = \{a^2_{\pi_2(z)}, a^2_{\pi_2(z+1)}, b_{m-2+\ell}, b_{m-1+\ell}\}$-CS, remove the LAVS or RAVS from (5.5) and apply an $\{a^2_{\pi_2(2)}, b_{m+\ell}, b_{m-2+\ell}\}$-RAVS. Similarly, if g_2 and h_2 are both red, define $X = \{a^2_{\pi_2(z)}, a^2_{\pi_2(z+1)}, b_{1+\ell}, b_{2+\ell}\}$-CS, remove the LAVS or RAVS from (5.5) and apply an $\{a^2_{\pi_2(2)}, b_{2+\ell}, b_{m+\ell}\}$-LAVS. Now, if there does not exist an integer $i \in \{-1, 0, 1, 2\}$, such that both g_i and h_i are red, then all vertical edges in both the $a^2_{\pi_2(z)}$ and $a^2_{\pi_2(z+1)}$-columns that are between the $b_{2+\ell}$ and $b_{m-2+\ell}$-rows are red. In this case, define $X = \{a^2_{\pi_2(z)}, a^2_{\pi_2(z+1)}, b_{3+\ell}, b_{4+\ell}\}$-CS, and remove the LAVS or RAVS from (5.5), and apply an $\{a^2_{\pi_2(2)}, b_{1+\ell}, b_{3+\ell}\}$-LAVS and an $\{a^2_{\pi_2(2)}, b_{4+\ell}, b_{m+\ell}\}$-LAVS. We now have a Hamilton decomposition of Γ. The case $z = 3$ follows similarly and is omitted.

(b) $\pi_2(z) \in \{1, 3\}$. If $t_2 = 2$, then $t_1 = 2$, and so the a^2_1 and a^2_3-columns are $\bigstar_2$-columns, which is resolved in Case 1.II(a). Hence, assume that $t_2 \geqslant 4$ and w.l.o.g., we may assume e_1 and e_2 are blue. Again, define g_i and h_i as in Case 1.II(a). The $a^2_{\pi_2(z)}$-column contains an *alternating path* of red and blue edges and as the $a^2_{\pi_2(z+1)}$-column contains at most two consecutive blue edges, there exists at least one integer $i \in \{-1, 0, 1-t_2, 2-t_2\}$ such that both g_i and h_i are red. Use the technique of Case 1.II(a), to define X.

(c) $\pi_2(z) = 2$. If $t_2 = 2$, then $t_1 = 2$, so that the a^2_2-column is a $\bigstar_2$-column, which is resolved in Case 1.II(a). Hence, assume that $t_2 \geqslant 4$ and assume that e_1 and e_2 are blue edges. First consider $m > t_2$ and let j be the smallest integer $1 \leqslant j \leqslant t_2 + 1$ such that both $(a^2_2, b_{j+\ell})(a^2_2, b_{j+1+\ell})$ and $(a^2_{\pi_2(z+1)}, b_{j+\ell})(a^2_{\pi_2(z+1)}, b_{j+1+\ell})$ are red edges. As there are at most $t_2 - 2$ blue edges in the a^2_2-column and at most two blue edges in the $a^2_{\pi_2(z+1)}$-column, such a j exists. Define

$$X := \{a^2_2, a^2_{\pi_2(z+1)}, b_{j+\ell}, b_{j+\ell+1}\}\text{-CS}$$

and remove the LAVS or RAVS of (5.5). Apply an

$$\{a^2_{\pi_2(2)}, b_{j+1+\ell}, b_{j+t_2-1+\ell}\}\text{-LAVS.}$$

As

$$j + t_2 - 1 + \ell \leqslant t_2 + 1 + t_2 - 1 + \ell = 2t_2 + \ell \leqslant m + \ell,$$

and e_3 is red, the aforementioned LAVS always defines a proper red/black color-switching configuration. By Remark 2.2.2, the result is a Hamilton decomposition. If $m = t_2$, then the a_2^2-column contains at most $m-2$ blue edges, and so it is possible that there exists no pair of edges $g = (a_2^2, b_{x+\ell})(a_2^2, b_{x+1+\ell})$ and $h = (a^2_{\pi_2(z+1)}, b_{x+\ell})(a^2_{\pi_2(z+1)}, b_{x+1+\ell})$ that are both red. However, this is only possible if no pair g and h are both blue, as there are at most two blue edges in the $a^2_{\pi_2(z+1)}$-column. In this case, we may remove the LAHS from (5.4) and apply the RAHS of (5.4) so that e_1 and e_3 are now blue. Clearly, the switch reflect does not change the position of the blue edges in the a_2^2-column, but it does shift the blue edges in the $a^2_{\pi_2(z+1)}$-column up or down one row, thereby creating a pair g and h that are *both red*. The conclusion now follows.

Case 2: One of t_1 and t_2 is even, the other is odd. W.l.o.g., let $t_2 \geqslant t_1$. The case $\overline{s_1} = \pm\overline{s_2}$ follows from the technique of Case 1.I. Hence, assume $\overline{s_1} \neq \pm\overline{s_2}$. If $t_2 = 2k_2 + 1 \geqslant 3$ and $t_1 = 2k_1 \geqslant 2$, then clearly $m \geqslant 2t_2 > t_2 > t_1$. Apply the same CS-configuration of (5.3) and (5.4) to $H_1 \cup F \simeq A_n^{(1)} \square_{r_1} B_m$, where we may assume $A_n^{(1)}$ and $A_n^{(2)}$ are the same as in Case 1.II. Following this technique, to $H_2 \cup F \simeq A^{(2)} \square_{r_2} B_m$, apply an

$$\{a^2_{\pi_2(2)}, b_{1+\ell}, b_{t_2+\ell}\}\text{-RAVS or -LAVS}$$

in place of the CS-configuration of (5.5). Define the color-switch X, between the $a^2_{\pi_2(z)}$ and $a^2_{\pi_2(z+1)}$-columns, as in Case 1.II. As $m > t_2$, we now have more freedom to define X. By Corollary 2.2.8, the application of X will produce a Hamilton decomposition of Γ.

Likewise, if $t_2 = 2k_2 \geqslant 4$ and $t_1 = 2k_1 + 1 \geqslant 3$, we may apply follow the technique of Case 1.II by applying an $\{a_2, b_1, b_{t_1}\}$-RAVS and an $\{a_3, a_4, b_{t_1}, b_{t_1+1}\}$-CS in place of (5.3) and an $\{a_4, a_{n-1}, b_{t_1}\}$-RAHS or $\{a_4, a_{n-1}, b_{t_1+1}\}$-LAHS in place of (5.4). By Corollary 2.2.8, the cycle structure is the same as in Case 1.II. We now have more freedom to define X, as $m > t_1$, and the result follows. ■

We now extend Lemma 5.2.7 to allow for $|s_3| = 4$.

Lemma 5.2.8. *If $\Gamma = \mathrm{C{\scriptsize AY}}(A, \{s_1, s_2, s_3\})$ is a connected, 6-regular, Cayley graph on A, where $|A : \langle s_3\rangle| \geqslant 9$, and $2s_1, 2s_2 \notin \langle s_3\rangle$, then Γ has a Hamilton decomposition.*

Proof. Note, $|s_3| \geqslant 3$, and by Lemmata 5.1.5(a) and 5.2.7, it suffices to prove the result when $|s_3| = 4$. W.l.o.g., $t_1 \geqslant t_2$. Similarly to Lemma 5.2.7, Γ is a $D(3, 4, n)$-graph with $H_1 \cup F \simeq A_n^{(1)} \square_{r_1} B_4$ and $H_1 \cup F \simeq A_n^{(2)} \square_{r_2} B_4$. W.l.o.g.,

$$A_n^{(1)} = a_1 a_2 a_3 \cdots a_p \cdots a_q \cdots a_n a_1,$$

$$A_n^{(2)} = a^2_{\pi_2(2)} a^2_{\pi_2(3)} a^2_{\pi_2(4)} \cdots a^2_{\pi_2(n)} a^2_p a^2_{\pi_2(2)} = c_1 c_2 c_3 \cdots c_n c_1,$$

where $\{\pi_2(2), \pi_2(3)\} = \{q, n\}$, and for simplicity of notation, we relabel the elements as $c_1, c_2, \ldots, c_n$. Apply the CS-configuration of Lemma 2.2.9 to $H_2 \cup F$, to obtain a black Hamilton cycle, and join the red edges in the c_1- and c_n-columns into one cycle. Apply the following CS-configuration to $A_n^{(1)} \square_{r_1} B_4$:

$$\begin{cases} \{a_{p-3}, a_{p-1}, b_2\}\text{-RAHS, } \{a_{p-1}, a_p, b_1, b_2\}\text{-CS} & \text{if } t_1 = 2 \\ \{a_{p-3}, a_{p-1}, b_2\}\text{-RAHS, } \{a_{p-1}, a_p, b_3, b_4\}\text{-CS} & \text{if } t_1 = 4 \end{cases} \tag{5.7}$$

In either case, a blue Hamilton cycle is created by Remark 2.2.2, and the a_i-columns, $p-3 \leqslant i \leqslant p-1$, are joined to the red cycle containing the $a^2_{\pi_2(2)}$ and a^2_p-columns.

Case 1. Let $n \geqslant 9$ odd. A red 2-factor now exists, consisting of $n-4$ cycles. We now closely follow the technique of the proof of Lemma 5.1.1 (Lemma 3.18 in [34]). Let $\{z_i : 1 \leqslant i \leqslant n-5\}$ be a sequence of integers satisfying $z_1 < z_2 < z_3 < \cdots < z_{n-5}$, such that, for $1 \leqslant i \leqslant n-5$, all vertical edges in the c_{z_i+1}-column are red. Clearly, $z_1 = 1$ and $z_{n-5} < n-1$. Consider color-switches, X_i, that are c-incident to the c_{z_i}- and c_{z_i+1}-columns. By Remark 2.2.2, applying $X_1, X_2, \ldots, X_{n-5}$ to $A_n^{(2)} \square_{r_2} B_4$, will create a red Hamilton cycle. Having already defined $X_1, X_2, \ldots, X_{2i-3}, X_{2i-2}$, define X_{2i-1} and X_{2i} as follows:

$$X_{2i-1} = \begin{cases} \{c_{z_{2i-1}}, c_{z_{2i-1}+1}, b_2, b_3\}\text{-CS} & \text{if } (c_{z_{2i-1}}, b_2)(c_{z_{2i-1}}, b_3) \text{ is red} \\ \{c_{z_{2i-1}}, c_{z_{2i-1}+1}, b_4, b_1\}\text{-CS} & \text{if } (c_{z_{2i-1}}, b_4)(c_{z_{2i-1}}, b_1) \text{ is red} \end{cases} \tag{5.8}$$

$$X_{2i} = \begin{cases} \{c_{z_{2i}}, c_{z_{2i}+1}, b_1, b_2\}\text{-CS} & \text{if } (c_{z_{2i}}, b_1)(c_{z_{2i}+1}, b_2) \text{ is red} \\ \{c_{z_{2i}}, c_{z_{2i}}, b_3, b_4\}\text{-CS} & \text{if } (c_{z_{2i}}, b_3)(c_{z_{2i}}, b_4) \text{ is red} \end{cases} \tag{5.9}$$

Clearly, by Remark 2.2.2, applying X_{2i-1} will break the black Hamilton cycle into two cycles, and applying X_{2i} will rejoin the black cycles again. Thus, Γ has a Hamilton decomposition.

Case 2. Let $n \geqslant 10$ even. We consider $\overline{s_1} \neq \pm\overline{s_2}$, for the case of equality is very similar and is omitted. By Remark 5.2.6, if two columns in $H_1 \cup F = A_n^{(1)} \square_{r_1} B_4$ are adjacent, they are not adjacent in $H_2 \cup F = A_n^{(2)} \square_{r_2} B_4$. Relocate the r_2-jump to be between the $a^2_{\pi_2(3)}$ and $a^2_{\pi_2(2)}$-columns, and relabel the columns of $A_n^{(2)} \square_{r_2} B_4$ by $c_1, c_2, \ldots, c_n$, with the c_1, c_{n-1}, and c_n-columns representing the $a^2_{\pi_2(3)}$, a^2_p, and $a^2_{\pi_2(2)}$-columns, respectively. To $H_1 \cup F$, apply an $\{a_{p-1}, a_p, b_1, b_2\}$-CS if $t_1 = 2$, or if $t_1 = 4$, apply the switching configuration

$$\{\{a_{p-3}, a_{p-2}, b_{1+\ell}, b_{2+\ell}\}\text{-CS}, \{a_{p-2}, a_{p-1}, b_{2+\ell}, b_{3+\ell}\}\text{-CS}, \{a_{p-1}, a_p, b_{3+\ell}, b_{4+\ell}\}\text{-CS}\},$$

to obtain a blue Hamilton cycle, and join the red edges in the involved columns into one cycle, where ℓ may vary depending on the context. To $H_2 \cup F$, apply the CS-configuration of Lemma 2.2.9, to create a black Hamilton cycle. Let c_{p_i} denote the a^2_{p-i}-column, where $i = 1, 2, 3$. There now exists a red 2-factor consisting of one cycle on the c_1 and c_n-columns, one cycle containing the c_{n-1}, c_{p_1}, c_{p_2}, and c_{p_3}-columns (or the c_{n-1} and c_{p_1}-columns if $t_1 = 2$), and $n-y$ cycles of length 4, each one a single column. Call these $n-y$ columns, *free columns*, where $y = 6$ if $t_1 = 4$, and $y = 4$ if $t_1 = 2$. Let the total number of red cycles be $2d$. Consider the integer sequence $\{z_i\}_{i=1}^{2d-1}$, where $1 = z_1 < z_2 < \cdots < z_{2d-1}$, such that the c_{z_i+1}-column is a free column or the left-most c_{p_i}-column. Let $\mathcal{X} = \{X_1, X_2, \cdots, X_{2d-1}\}$ be a set of edge-disjoint color-switches where X_i is c-incident to the c_{z_i}- and c_{z_i+1}-columns. After applying $\mathcal{X}$ to $H_2 \cup F$, we have a red Hamilton cycle. However, the black Hamilton cycle is now split into two cycles. If the red Hamilton cycle is given a direction, all edges in a fixed column have the same direction, edges in the c_{z_i}- and c_{z_i+1}-columns have opposite direction, and edges in the c_1- and c_n-columns have the same direction. As n is even, there exists an integer z, such that the edges in the c_z and c_{z+1}-columns have the same direction. Applying any color-switch, X', between those two columns, will preserve the red Hamilton cycle. Insert z into $\{z_i\}_{i=1}^{2d-1}$, and X' into $\mathcal{X}$, and relabel, so that we have $1 \leqslant z'_1 < z'_2 < \cdots < z'_{2d} < n$ and $\mathcal{X} = \{X_1, X_2, \ldots, X_{2d}\}$. Clearly, $z \neq z_i$, and so the c_{z+1}-column is not a free column. We now construct the set $\mathcal{X}$. Having already defined $X_1, X_2, \ldots, X_{2i-3}, X_{2i-2}$, we now define X_{2i-1} and X_{2i}.

1. $z \notin \{z'_{2i-1}, z'_{2i}, 1, n-1\}$. Let

$$X_{2i-1} := \begin{cases} \{c_{z'_{2i-1}}, c_{z'_{2i-1}+1}, b_2, b_3\}\text{-CS} & \text{if } (c_{z'_{2i-1}}, b_2)(c_{z'_{2i-1}}, b_3) \text{ is red,} \\ \{c_{z'_{2i-1}}, c_{z'_{2i-1}+1}, b_4, b_1\}\text{-CS} & \text{if } (c_{z'_{2i-1}}, b_4)(c_{z'_{2i-1}}, b_1) \text{ is red.} \end{cases}$$

and

$$X_{2i} := \begin{cases} \{c_{z'_{2i}}, c_{z'_{2i}+1}, b_3, b_4\}\text{-CS} & \text{if } (c_{z'_{2i}}, b_3)(c_{z'_{2i}}, b_4) \text{ is red,} \\ \{c_{z'_{2i}}, c_{z'_{2i}+1}, b_1, b_2\}\text{-CS} & \text{if } (c_{z'_{2i}}, b_1)(c_{z'_{2i}}, b_2) \text{ is red.} \end{cases}$$

2. $z \in \{z'_{2i-1}, z'_{2i}\}$, but $z \notin \{1, n-1\}$. We consider two subcases: (1) where c_z is a free column and (2) where c_z is one of the c_{p_i}-columns. First, let the c_z-column be a free column. Now the c_{z+1}-column is one of the three c_{p_i}-columns that is not the left-most one, and has at most two consecutive non-red edges. If $z = z'_{2i-1}$, then as both $(c_z, b_2)(c_z, b_3)$ and $(c_z, b_4)(c_z, b_1)$ are red, define $X_{2i-1} := \{c_z, c_{z+1}, b_2, b_3\}$-CS or $X_{2i-1} := \{c_z, c_{z+1}, b_4, b_1\}$-CS, depending on which of $(c_{z+1}, b_2)(c_{z+1}, b_3)$ or $(c_{z+1}, b_4)(c_{z+1}, b_1)$ is red. Similarly, if $z = z'_{2i}$, then as both $(c_z, b_1)(c_z, b_2)$ and $(c_z, b_3)(c_z, b_4)$ are red, define $X_{2i} := \{c_z, c_{z+1}, b_1, b_2\}$-CS or $X_{2i} := \{c_z, c_{z+1}, b_3, b_4\}$-CS, depending on which of $(c_{z+1}, b_1)(c_{z+1}, b_2)$ or $(c_{z+1}, b_3)(c_{z+1}, b_4)$ is red.

   Now suppose the c_z-column is the c_{p_3}-column. By Lemma 5.2.5, the c_{z+1}-column can be either the c_{n-1} or c_{p_1}-column. Consider

   $$e_{f+i} := (c_z, b_{1+i})(c_z, b_{2+i}) \text{ and } g_{f+i} := (c_{z+1}, b_{1+i})(c_{z+1}, b_{2+i}).$$

   If $z = z'_{2i-1}$, choose ℓ in the switching configuration of $H_1 \cup F$, so that either e_{f+1} and g_{f+1} are red, or e_{f+3} and g_{f+3} are red, depending on if the color switch, X_{2i-2} was r-incident to the b_1 and b_2-rows or the b_3 and b_4-rows. Define X_{2i-1} to be the corresponding switch r-incident to either the b_1 and b_2-rows, or the b_4 and b_1-rows. The result is obtained similarly for $z = z'_{2i}$ by choosing ℓ so that either e_f and g_f are red or e_{f+2} and g_{f+2} are red. Suppose the c_z-column is the c_{p_2}-column. Hence, by Lemma 5.2.5, the c_{z+1}-column must be the c_{n-1}-column. As the c_z-column is not the left-most column in its cycle, it has exactly two non-red edges. Therefore, if $z = z'_{2i-1}$, we may choose ℓ so that either $(c_z, b_2)(c_z, b_3)$ and $(c_z, b_4)(c_z, b_1)$ are red or $(c_{n-1}, b_2)(c_{n-1}, b_3)$ and $(c_{n-1}, b_4)(c_{n-1}, b_1)$ are red, and let X_{2i-1} be the corresponding switch. The result is obtained similarly for $z = z'_{2i}$. If the c_z-column is the c_{p_1}-column, then by Lemma 5.2.5, the c_{z+1}-column must be the c_{p_3}-column, and result follows similarly.

3. $z \in \{1, n-1\}$. If $z = 1$, define the color switch $X_1 := \{c_1, c_2, b_2, b_3\}$-CS or $X_1 := \{c_1, c_2, b_4, b_1\}$-CS depending which forms a good switch. Likewise, if $z = n-1$, then define the color switch $X_{2d} := \{c_{n-1}, c_n, b_1, b_2\}$-CS or $X_{2d} := \{c_{n-1}, c_n, b_3, b_4\}$-CS depending on which forms a good switch. ■

Lemmata 5.2.7 and 5.2.8 combine to yield the following.

Theorem 5.2.9. *If A is an abelian group, and for some $1 \leqslant i \leqslant 3$, $\mathrm{Cay}(A/\langle s_i \rangle, \{\overline{s_{j_1}}, \overline{s_{j_2}}\})$ is 4-regular and of order at least nine, then $\mathrm{Cay}(A, \{s_1, s_2, s_3\})$ has a Hamilton decomposition.*

5.3 Decompositions for Low-Order Quotient Graphs

In this section, we consider Case (h) of Remark 5.1.6. The technique used in the proofs of Lemmata 5.3.1 and 5.3.3 is similar to that used in the proof of Theorem 3.2.1.

5.3.1 Odd-order quotients

Lemma 5.3.1. *If A is an abelian group, and for some $1 \leqslant i \leqslant 3$, $\textsc{Cay}(A/\langle s_i\rangle, \{\overline{s_{j_1}}, \overline{s_{j_2}}\})$ is 4-regular and of order $n \in \{1, 5, 7\}$, then $\textsc{Cay}(A, \{s_1, s_2, s_3\})$ has a Hamilton decomposition.*

Proof. Let $[A : \langle s_i\rangle] = n$. If $n = 1$, then s_i generates A, a case resolved by Theorem 1.5.2. By assumption, $\overline{s_j} \neq \overline{0}$, so $|\overline{s_j}| \geqslant 3$. Let $\langle \overline{s_{j_1}}, \overline{s_{j_2}}\rangle = A/\langle s_i\rangle$, where $|s_{j_1}| \geqslant |s_{j_2}|$ and $|\langle s_i\rangle| = m$. If $m = 2j + 1$, then $|A|$ is odd, a case resolved by Theorem 3.2.1. Hence, without loss of generality, $m = 2d \geqslant 4$. As n is prime, $|\overline{s_{j_i}}| = n$ for $i = 1, 2$. The quotient graph, $\Delta = \textsc{Cay}(A/J, \{\overline{s_{j_1}}, \overline{s_{j_2}}\})$ is 4-regular and connected. Thus, by Theorem 1.5.1, Δ has a Hamilton decomposition into two cycles $\overline{H_1}$ and $\overline{H_2}$, where $\overline{H_i}$ is just the cycle generated by $\overline{s_{j_i}}$. The lift of $\overline{H_i}$, denoted H_i, is the 2-factor generated by s_{j_i} in Γ. By Theorem 3.1.6, Γ is a $D(3, m, n)$-graph, where F is the red 2-factor generated by s_i. Thus $H_i \cup F = A_n^{(i)} \square_{r_i} B_m$, for $i = 1, 2$, where

$$A_n^{(i)} := a^i_{\pi_i(1)} a^i_{\pi_i(2)} \cdots a^i_{\pi_i(n)} a^i_{\pi_i(1)}.$$

Note, for $i = 1, 2$, H_i consists of $t_i = \gcd(r_i, m) = |A : \langle s_{j_i}\rangle|$ cycles of length $|s_{j_i}| = nm/t_i$. If $t_i = 1$ for some $i = 1, 2$, then s_{j_i} generates A, a case resolved by Theorem 1.5.2. Furthermore, if $t_1 = 2$, then we are done by Corollary 4.1.6. Thus, assume $m \geqslant t_2 \geqslant t_1 \geqslant 3$.

Case 1: $n = 5$. Up to relabeling, Δ is either, $\Lambda_1 := \textsc{Cay}(\mathbb{Z}_5, \{\pm 1\})$, a multigraph on five vertices, or $\Lambda_2 := \textsc{Cay}(\mathbb{Z}_5, \{1, 2\}) = K_5$. If $\Delta = \Lambda_1$, without loss of generality, $\pi_1 = \pi_2 = (1)$. If $\Delta = \Lambda_2$, then without loss of generality, $\pi_1 = (1)$ and $\pi_2 = (2354)$ so that $A_5^{(2)} := a_1^2 a_3^2 a_5^2 a_2^2 a_4^2 a_1^2$. The corresponding quotient graphs are shown in Figure 3.3 of Chapter 3.

1.i. $\boldsymbol{t_2 = 2k_2 \geqslant 2k_1 = t_1 \geqslant 4}$. Apply the CS-configuration of Lemma 2.2.9 to $H_1 \cup F$ to obtain a blue Hamilton cycle and join the vertical red edges in the a_1 and a_5-columns into one cycle C.

   If $\Delta = \Lambda_1$, let ℓ be any integer such that $(a_1^2, b_{1+\ell})(a_2^2, b_{2+\ell})$ is blue, and apply a switching configuration of Theorem 2.2.7(b), using a RAVS with $i = 2$ to obtain a set of three monochromatic Hamilton cycles in Γ.

   If $\Delta = \Lambda_2$ then let $(a_1^2, b_{1+\ell})(a_2^2, b_{2+\ell})$ be blue as before, and apply an $\{a_3^2, b_{1+\ell}, b_{t_2-1+\ell}\}$-RAVS and either an $\{a_2^2, a_4^2, b_{t_2-1+\ell}, b_{t_2+\ell}\}$-CS or an $\{a_2^2, a_4^2, b_{m+\ell}, b_{1+\ell}\}$-CS, depending on which of $(a_2^2, b_{t_2-1+\ell})(a_2^2, b_{t_2+\ell})$ or $(a_2^2, b_{1+\ell})(a_2^2, b_{m+\ell})$ is red. The result is a Hamilton decomposition of Γ.

1.ii. $\boldsymbol{t_2 = 2k_2 > 2k_1 + 1 = t_1 \geqslant 3}$. As t_2 is even, apply the CS-configuration of Lemma 2.2.9 to $H_2 \cup F$ to obtain a black Hamilton cycle.

   If $\Delta = \Lambda_1$, this joins all vertical red edges in the a_1^2 and a_5^2-columns into one cycle. Let ℓ be the integer such that $(a_1, b_{1+\ell})(a_1, b_{2+\ell})$ is a black edge. As the red and black edges in the a_1-column form a matching, apply an $\{a_2, b_{1+\ell}, b_{t_1+\ell}\}$-RAVS to obtain a blue Hamilton cycle and join all vertical red edges in the a_i-columns, where $i \in \{1, 2, 3, 5\}$, into one cycle. Finally, by Theorem 2.2.7, applying an $\{a_3, a_4, b_{t_1+\ell}, b_{t_1+1+\ell}\}$-CS preserves the blue cycle and produces a red Hamilton cycle. The result is a Hamilton decomposition of Γ.

   If $\Delta = \Lambda_2$, the color switching configuration of Lemma 2.2.9 joins all vertical red edges in the a_1^2 and a_4^2-columns into one cycle. Let ℓ be the integer such that $(a_1, b_{1+\ell})(a_1, b_{2+\ell})$ is black. Apply an $\{a_2, b_{1+\ell}, b_{t_1+\ell}\}$-RAVS, which by Theorem 2.2.7, generates a blue Hamilton cycle and joins all vertical red edges in the a_i-columns, where $1 \leqslant i \leqslant 4$, into one cycle. The vertical red edges in the a_4-column, which also form a matching with black edges, have the property that either $(a_4, b_{t_1+\ell})(a_4, b_{t_1+1+\ell})$ is red, or $(a_4, b_{m+\ell})(a_4, b_{1+\ell})$ is red. Apply either an $\{a_4, a_5, b_{t_1+\ell}, b_{t_1+1+\ell}\}$-CS or an $\{a_4, a_5, b_{m+\ell}, b_{1+\ell}\}$-CS, accordingly. By Corollary 2.2.8, this creates three monochromatic Hamilton cycles in Γ.

1.iii. $\boldsymbol{t_2 = 2k_2 + 1 > 2k_1 = t_1 \geqslant 4}$. This case follows identically to 2.ii. by applying the CS-configuration of Lemma 2.2.9 to $H_1 \cup F$ rather than $H_2 \cup F$.

1.iv. $\boldsymbol{t_2 = 2k_2 + 1 \geqslant 2k_1 + 1 = t_1 \geqslant 3}$.

If $\Delta = \Lambda_1$, then apply an $\{a_2, b_1, b_{t_1}\}$-LAVS to obtain a blue Hamilton cycle and join the vertical red edges in the a_i-columns, where $1 \leqslant i \leqslant 3$, into one cycle C. As the vertical red and blue edges in the a_3-column form a matching, let ℓ be the integer such that $(a_3^2, b_{1+\ell})(a_3^2, b_{2+\ell})$ is red. Apply an $\{a_4^2, b_{1+\ell}, b_{t_2+\ell}\}$-LAVS to obtain a black Hamilton cycle and join the vertical red edges in the a_4^2 and a_5^2-columns to C. The result is a Hamilton decomposition of Γ.

If $\Delta = \Lambda_2$, then $m \geqslant 2t_2$. Apply an $\{a_2^2, b_1, b_{t_2}\}$-LAVS to obtain a black Hamilton cycle and join all vertical red edges in the $a^2_{\pi_2(i)}$-columns, where $3 \leqslant i \leqslant 5$, into one cycle C. Let ℓ be the integer such that all vertical edges in the a_2-column that are between the $b_{1+\ell}$ and $b_{t_2+\ell}$-rows are black. By our assumption on m, all vertical edges in the a_2-column between the $b_{t_2+\ell}$ and $b_{t_2+t_1-1+\ell}$-rows are red. Thus, apply an $\{a_2, b_{t_2+\ell}, b_{t_2+t_1-1+\ell}\}$-RAVS to obtain a blue Hamilton cycle, and join the vertical red edges in the a_1 and a_2-columns to C. The result is a Hamilton decomposition of Γ.

Case 2: $\boldsymbol{n = 7}$. Again, up to relabeling, Δ is either $\Lambda_1 := \textsc{Cay}(\mathbb{Z}_7, \{\pm 1\})$, $\Lambda_2 := \textsc{Cay}(\mathbb{Z}_7, \{1, 3\})$, or $\Lambda_3 := \textsc{Cay}(\mathbb{Z}_7, \{1, 2\})$. If $\Delta = \Lambda_1$, then without loss of generality, $\pi_1 = \pi_2 = (1)$, if $\Delta = \Lambda_2$, then without loss of generality, $\pi_1 = (1)$, $\pi_2 = (243756)$ and so

$$A_7^{(2)} := a_1^2 a_4^2 a_7^2 a_3^2 a_6^2 a_2^2 a_5^2 a_1^2.$$

If $\Delta = \Lambda_3$, then without loss of generality, $\pi_1 = (1)$, $\pi_2 = (235)(476)$ and

$$A_7^{(2)} := a_1^2 a_3^2 a_5^2 a_7^2 a_2^2 a_4^2 a_6^2 a_1^2.$$

The corresponding quotient graphs are shown in Figure 3.4 of Chapter 3.

2.i. $\boldsymbol{t_2 = 2k_2 \geqslant 2k_1 = t_1 \geqslant 4}$.

If $\Delta = \Lambda_1$, then apply the CS-configuration of Lemma 2.2.9 to $H_1 \cup F$ to obtain a blue Hamilton cycle and join the a_1 and a_7-columns into one red cycle C. Apply an $\{a_2^2, b_{1+\ell}, b_{t_2-1+\ell}\}$-RAVS, where ℓ is chosen as before. By Theorem 2.2.7(b), applying an $\{a_2^2, a_3^2, b_{t_2-1+\ell}, b_{t_2+\ell}\}$-CS will produce a black Hamilton cycle and join the vertical red edges in the a_i^2-columns, where $i = 2, 3, 4$ to the cycle C. Finally, by Lemma 2.2.15, applying an $\{a_4^2, a_6^2, b_{t_2-1+\ell}\}$-RAHS will preserve the black cycle, and create three monochromatic Hamilton cycles in Γ.

If $\Delta = \Lambda_2$, then apply a $\{a_4^2, b_1, b_{t_2-1}\}$-LAVS and a $\{a_7^2, a_3^2, b_{t_2-1}, b_{t_2}\}$-CS. The result is a black Hamilton cycle and a red 2-factor consisting of one cycle, call it C, on the vertical red edges in the a_i^2-columns, where $i = 1, 4, 7, 3$, and three remaining cycles on the a_6^2, a_2^2, and a_5^2-columns, respectively. Let x be the integer such that the edge $(a_7, b_{1+x})(a_7, b_{2+x})$ is non-red. As the red and non-red edges in the a_7-column form a matching, apply a $\{a_6, b_{1+x}, b_{t_1-1+x}\}$-LAVS to join all vertical red edges in the a_5 and a_6-columns to C. We now have a red 2-factor consisting of the red vertical edges in the a_i-columns, where $i \neq 2$, and a red m-cycle on the a_2-column. If $t_1 < m$, then one of the edges $e := (a_3, b_{m+x})(a_3, b_{1+x})$ or $e' := (a_3, b_{t_1-1+x})(a_3, b_{t_1+x})$ are red.

Apply either a $\{a_2, a_3, b_{m+x}, b_{1+x}\}$-CS or a $\{a_2, a_3, b_{t_1-1+x}, b_{t_1+x}\}$-CS, accordingly to obtain a Hamilton decomposition. If $t_1 = m$, then $e = e'$. If e is red, we are done. Otherwise, e is the only non-red edge in the a_3-column, and we may replace x with $x+2$, to obtain the result.

If $\Delta = \Lambda_3$, then apply a $\{a_2, b_1, b_{t_1-1}\}$-RAVS and a $\{a_3, a_4, b_{t_1-1}, b_{t_1}\}$-CS to obtain a blue Hamilton cycle and join the vertical red edges in the a_i-columns, where $1 \leqslant i \leqslant 4$ into one cycle, C. The red and non-red edges in the a_3-column form a matching. Thus, let ℓ be the integer such that the edge $(a_3^2, b_{1+\ell})(a_3^2, b_{2+\ell})$ is non-red. As $(a_3^2, b_{2+\ell})(a_3^2, b_{3+\ell})$ must be red, apply a $\{a_5^2, b_{1+\ell}, b_{t_2-1+\ell}\}$-RAVS to connect all vertical red edges in the a_i-columns, where $i = 1, 2, 3, 4, 5, 7$ into one cycle. The red edges in the a_6^2-column lie in their own m-cycle. Let $e := (a_4^2, b_{m+\ell})(a_4^2, b_{1+\ell})$ and $f := (a_4^2, b_{t_2-1+\ell})(a_4^2, b_{t_2+\ell})$. If $t_2 < m$, then $e \neq f$. Thus, apply a $\{a_4^2, a_6^2, b_{m+\ell}, b_{1+\ell}\}$-CS if e is red, or a $\{a_4^2, a_6^2, b_{t_2-1+\ell}, b_{t_2+\ell}\}$-CS otherwise. As the black horizontal edges in the $b_{m+\ell}$ and $b_{t_2+\ell}$-rows lie on the same cycle, and the black edges in the $b_{1+\ell}$ and $b_{t_2-1+\ell}$-rows lie together on a different cycle, the appropriately chosen color switch will create Hamilton decomposition of Γ. If $t_2 = m$, then $e = f$. If e is red, proceed as before to obtain the result. If e is non-red, then clearly $(a_4^2, b_{2+\ell})(a_4^2, b_{3+\ell})$ is red. Apply a $\{a_5^2, b_{3+\ell}, b_{1+\ell}\}$-RAVS rather than the $\{a_5^2, b_{1+\ell}, b_{t_2-1+\ell}\}$-RAVS (i.e. replace ℓ with $\ell+2$) to obtain the result.

2.ii. $\boldsymbol{t_2 = 2k_2 > 2k_1 + 1 = t_1 \geqslant 3}$.

If $\Delta = \Lambda_1$, then apply an $\{a_2, b_1, b_{t_1}\}$-RAVS and an $\{a_3, a_4, b_{t_1}, b_{t_1+1}\}$-CS. By Corollary 2.2.15(a), the result is a blue Hamilton cycle and a red cycle on the a_i-columns, where $1 \leqslant i \leqslant 4$. Choose ℓ so that $(a_4^2, b_{1+\ell})(a_4^2, b_{2+\ell})$ is red, and apply an $\{a_5^2, b_{1+\ell}, b_{t_2-1+\ell}\}$-LAVS and an $\{a_6^2, a_7^2, b_{t_2-1+\ell}, b_{t_1+\ell}\}$-CS to obtain three monochromatic Hamilton cycles.

If $\Delta = \Lambda_2$, apply a $\{a_2, b_1, b_{t_1}\}$-RAVS to obtain a blue Hamilton cycle and join the a_i-columns into one red cycle, where $1 \leqslant i \leqslant 3$. By Corollay 2.2.8, applying an $\{a_5, a_6, b_{t_1}, b_{t_1+1}\}$-CS will preserve the blue cycle, and join the a_5 and a_6-columns into one red cycle. As $m \geqslant 6$, we can find an integer ℓ such that the edge $(a_3^2, b_{1+\ell})(a_3^2, b_{2+\ell})$ is non-red and the edge $(a_6^2, b_{m+\ell})(a_6^2, b_{1+\ell})$ is red. Note the red and non-red edges in the a_3^2-column form a matching. Apply a $\{a_7^2, b_{1+\ell}, b_{t_2-1+\ell}\}$-LAVS to join the $a^2_{\pi_2(i)}$-columns into one red cycle, where $i \in \{1, 2, 3, 4, 6\}$. Clearly, the black edges in the $b_{m+\ell}$-row are on a different cycle than the black edges in the $b_{j+\ell}$-rows, where $1 \leqslant j \leqslant t_2 - 1$. Furthermore, the a_3^2 and a_6^2-columns are on two different red cycles. Thus, by Corollary 2.2.8(b), apply an $\{a_3^2, a_6^2, b_{m+\ell}, b_{1+\ell}\}$-CS to obtain a Hamilton decomposition.

If $\Delta = \Lambda_3$, then apply the CS-configuration of Theorem 2.2.7(a) using a RAVS and $\ell = 0$ to obtain a blue Hamilton cycle. Choose x so that $(a_3^2, b_{1+x})(a_3^2, b_{2+x})$ is blue, and apply an $\{a_5^2, b_{1+x}, b_{t_2-1+x}\}$-RAVS. Now, there exists only one non-red edge in the a_4^2-column and so one of $(a_4^2, b_{t_2-1+x})(a_4^2, b_{t_2+x})$ or $(a_4^2, b_{m+x})(a_4^2, b_{1+x})$ is red. Apply an appropriate color-switch c-incident to the a_4^2 and a_6^2-columns, which by Corollary 2.2.8(b), yields a set of three monochromatic Hamilton cycles.

2.iii. $\boldsymbol{t_2 = 2k_2 + 1 > 2k_1 = t_1 \geqslant 4}$.

If $\Delta = \Lambda_1$, apply an $\{a_2, b_1, b_{t_1-1}\}$-RAVS and an $\{a_3, a_4, b_{t_1-1}, b_{t_1}\}$-CS. The result is a blue Hamilton cycle and a red cycle consisting of the vertical red edges in the a_i-columns, where $1 \leqslant i \leqslant 4$. Choose ℓ so that $(a_4^2, b_{1+\ell})(a_4^2, b_{2+\ell})$ is blue. Apply the CS-configuration of Theorem 2.2.7(a) using an $\{a_5^2, b_{1+\ell}, b_{t_2+\ell}\}$-RAVS to obtain three monochromatic Hamilton cycles.

If $\Delta = \Lambda_2$, then apply an $\{a_2, b_1, b_{t_1-1}\}$-RAVS and an $\{a_3, a_4, b_{t_1-1}, b_{t_1}\}$-CS to create a blue Hamilton cycle and a matching of red and blue edges in the a_3-column and a path of $t_1 - 2$ blue vertical edges in the a_2-column. Choose x so that all vertical edges in the a_3^2-column between the b_{1+x} and b_{t_2+x}-rows are red. As $m \geqslant 2t_2$, such a path exists. Apply an $\{a_3^2, b_{1+x}, b_{t_2+x}\}$-RAVS to obtain a black Hamilton cycle. At this point, we have a red 2-factor consisting of all vertical red edges in the a_i^2-columns, where $i = 1, 2, 3, 4, 5, 6$. Now, as $t_2 > t_1$, at least one of the edges $(a_2^2, b_{m+x})(a_2^2, b_{1+x})$ and $(a_2^2, b_{t_2+x})(a_2^2, b_{t_2+1+x})$ is red. Call this red edge e. By Corollary 2.2.8, we may apply one color switch that is c-incident to e and the a_5^2-column. The result is a Hamilton decomposition of Γ. The case $\Delta = \Lambda_3$ follows similarly.

2.iv. $\boldsymbol{t_2 = 2k_2 + 1 \geqslant 2k_1 + 1 = t_1 \geqslant 3}$.

If $\Delta = \Lambda_1$, then apply an $\{a_2, b_1, b_{t_1}\}$-RAVS and an $\{a_3, a_4, b_{t_1}, b_{t_1+1}\}$-CS to obtain a blue Hamilton cycle by Theorem 2.2.7(a), and join the vertical red edges in the a_i-columns, where $1 \leqslant i \leqslant 4$, into one red cycle. Choose ℓ so that $(a_4^2, b_{2+\ell})(a_4^2, b_{3+\ell})$ is red, and apply an $\{a_5^2, b_{1+\ell}, b_{t_2+\ell}\}$-RAVS and an

$$\{a_6^2, a_7^2, b_{t_2+\ell}, b_{t_2+1+\ell}\}\text{-CS},$$

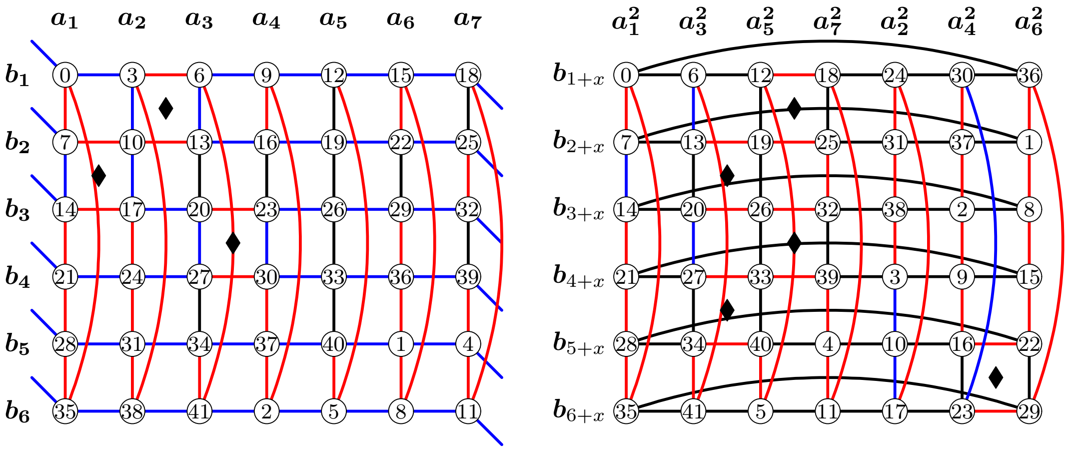
which by Theorem 2.2.7(a), produces three monochromatic Hamilton cycles.

If $\Delta = \Lambda_2$, then apply an $\{a_2, b_1, b_{t_1}\}$-LAVS to obtain a blue Hamilton cycle. By Lemma 2.2.15, applying an $\{a_3, a_5, b_2\}$-RAHS will preserve the blue cycle and join the vertical red edges in the a_i-columns, where $1 \leqslant i \leqslant 5$ into one red cycle, C. As $m \geqslant 2t_2 \geqslant 2t_1$, we can choose x so that all vertical edges between the b_{1+x} and b_{t_2+x}-rows are red. Then, apply an $\{a_3^2, b_{1+x}, b_{t_2+x}\}$-LAVS or -RAVS to produce a black Hamilton cycle and join the vertical red edges in the a_6^2 and a_7^2-columns to C. The result is a Hamilton decomposition of Γ.

If $\Delta = \Lambda_3$, then apply an $\{a_2, b_1, b_{t_1}\}$-LAVS to obtain a blue Hamilton cycle. As $m \geqslant 2t_2 \geqslant 2t_1$, we can choose x so that all vertical edges between the b_{1+x} and b_{t_1+x}-rows are blue. Hence, the vertical edges between the b_{t_1+x} and $b_{t_1+t_2-1+x}$-rows forms a red path of length $t_2 - 1$. Apply an $\{a_7^2, b_{t_1+x}, b_{t_1+t_2-3+x}\}$-LAVS and either an $\{a_4^2, b_{t_1+t_2-3+x}, b_{t_1+t_2-1+x}\}$-RAVS or -LAVS to create a black Hamilton cycle and join all vertical red edges into a single cycle, for a Hamilton decomposition of Γ. ■

Example 5.3.2. Consider the graph $\text{Cay}(\mathbb{Z}_{42}, \{s_1 = 3, s_2 = 6, s_3 = 7\})$ from Example 3.1.7. The labeling that was applied allows us to view the quotient graph $\Delta = \text{Cay}(\mathbb{Z}_{42}/\langle 7\rangle, \{\overline{3}, \overline{6}\})$ as $\Lambda_3 = \text{Cay}(\mathbb{Z}_7, \{1, 2\})$. Furthermore, $t_2 = 6 > t_1 = 3$, and by Case 2.ii with $n = 7$ in Lemma 5.3.1, we may apply an $\{a_2, b_1, b_3\}$-RAVS and an $\{a_3, a_4, b_3, b_4\}$-CS to $A_7 \square_3 B_6$ to obtain a blue Hamilton cycle and join the red cycles in the a_i-columns, where $1 \leqslant i \leqslant 4$ together. Next, we choose $x = 0$, so that $(a_3^2, b_1)(a_3^2, b_2)$ is a blue edge and apply an $\{a_5^2, b_1, b_5\}$-RAVS and an $\{a_4^2, a_6^2, b_5, b_6\}$-CS to create three monochromatic Hamilton cycles, i.e., a Hamilton decomposition, $\{D_1, D_2, D_3\}$, of $\text{Cay}(\mathbb{Z}_{42}, \{3, 6, 7\})$ (see Figure 5.1).

$$\begin{aligned} D_1 &= 0, 3, 10, 17, 20, 27, 24, 21, 18, 15, 12, 9, 6, 13, 16, 19, 22, 25, 28, 31, 34, 37, 40, 1, 4, 7, 14, 11, 8, 5, \\ &\quad\; 2, 41, 38, 35, 32, 29, 26, 23, 30, 33, 36, 39 \\ D_2 &= 0, 7, 10, 13, 19, 25, 32, 26, 20, 23, 29, 36, 1, 8, 15, 22, 16, 9, 2, 37, 30, 27, 33, 39, 4, 11, 18, 12, 5, 40, \\ &\quad\; 34, 41, 6, 3, 38, 31, 24, 17, 14, 21, 28, 35 \\ D_3 &= 0, 6, 12, 19, 26, 33, 40, 4, 10, 16, 23, 17, 11, 5, 41, 35, 29, 22, 28, 34, 27, 21, 15, 9, 3, 39, 32, 38, 2, 8, \\ &\quad\; 14, 20, 13, 7, 1, 37, 31, 25, 18, 24, 30, 36 \end{aligned}$$



$H_1 \cup F = A_7 \,\square_3\, B_6$ $\qquad$ $H_2 \cup F = A_7^{(2)} \,\square\, B_6$

Figure 5.1: A Hamilton decomposition of $\textsc{Cay}(\mathbb{Z}_{42}, \{3, 6, 7\}) = H_1 \cup H_2 \cup F$, from Example 5.3.2 with $x = 0$. (A "♦" represents a color switch on the 4-cycle surrounding it.)

5.3.2 Even-order quotients

Lemma 5.3.3. *If A is an abelian group, and for some $1 \leqslant i \leqslant 3$, $\textsc{Cay}(A/\langle s_i\rangle, \{\overline{s_{j_1}}, \overline{s_{j_2}}\})$ is 4-regular and of order $n \in \{4, 6, 8\}$, then $\textsc{Cay}(A, \{s_1, s_2, s_3\})$ has a Hamilton decomposition.*

Proof. As in Lemma 5.3.1, we have $|\overline{s_{j_1}}|, |\overline{s_{j_2}}| \geqslant 3$, where, without loss of generality, $|s_{j_1}| \geqslant |s_{j_2}| \geqslant 3$ and $|\langle s_i\rangle| = m \geqslant 3$. By Theorem 3.1.6, Γ is a $D(3, m, n)$-graph, where F is the red 2-factor generated by s_i, and $H_i \cup F = A_n^{(i)} \,\square_{r_i}\, B_m$, for $i = 1, 2$. If $t_i = 1$ for some $i = 1, 2$, then H_i is itself a Hamilton cycle, and upon deletion, leaves a pseudo-cartesian product of cycles, which is Hamilton decomposable by Theorem 1.5.1. Thus, assume $t_2 \geqslant t_1 \geqslant 2$.

Case 1: $\boldsymbol{n = 4}$. Clearly, $|\overline{s_{j_1}}| = |\overline{s_{j_2}}| = 4$, thus $A/J \cong \mathbb{Z}_4$, i.e., $\Delta = \textsc{Cay}(\mathbb{Z}_4, \{\pm 1\})$, as unlabeled graphs. Without loss of generality, $\pi_1 = \pi_2 = (1)$ and H_i is just the 2-factor generated by s_{j_i} for $i = 1, 2$. If $m = 3$, then $|A| = 12$, and so $A \cong \mathbb{Z}_{12}$ or $A \cong \mathbb{Z}_2 \times \mathbb{Z}_6$. Hamilton decompositions for all non-isomorphic, 6-regular, connected Cayley graphs on theses groups are given in the Appendix in Table 1.1. Thus, assume $m \geqslant 4$ and, by Corollary 4.1.6, assume $t_1 \geqslant 3$.

1.i. $\boldsymbol{t_2 = 2k_2 \geqslant 2k_1 = t_1 \geqslant 4}$. Apply the CS-configuration of Lemma 2.2.9 to $H_1 \cup F$ to obtain a blue Hamilton cycle and join the red edges in the a_1^2 and a_4^2-columns. Choose x so that $(a_1^2, b_{1+x})(a_1^2, b_{2+x})$ is a red edge. Apply an $\{a_2^2, b_{1+x}, b_{t_2-1+x}\}$-LAVS to create a red Hamilton cycle. Make the red cycle a directed cycle. By Lemma 2.2.9, all vertical red edges in the a_1^2 and a_4^2-columns share the same orientation, call them $\uparrow$-edges. The application of the LAVS forces the red edges in the a_2^2-column to be $\downarrow$-edges, and the red edges in the a_3^2-column to be $\uparrow$-edges. Thus, we may apply a color-switch, call it $\mathcal{X}$, that is c-incident to the a_3^2 and a_4^2-columns, and preserve the red Hamilton cycle. As the vertical edges in the a_4^2-column form a matching of red and blue edges, exactly one of $e := (a_4^2, b_{m+x})(a_4^2, b_{1+x})$ or $f := (a_4^2, b_{t_2-1+x})(a_4^2, b_{t_2+x})$ is red. Therefore, define $\mathcal{X}$ as follows:

$$\mathcal{X} := \begin{cases} \{a_3^2, a_4^2, b_{t_2-1+x}, b_{t_2+x}\}\text{-CS} & \text{if } f \text{ is red.} \\ \{a_3^2, a_4^2, b_{m+x}, b_{1+x}\}\text{-CS} & \text{if } f \text{ is non-red.} \end{cases}$$

The result is a set of three monochromatic Hamilton cycles.

1.ii. $\boldsymbol{t_2 = 2k_2 > 2k_1 + 1 = t_1 \geqslant 3}$. Apply the CS-configuration of Lemma 2.2.9 to $H_2 \cup F$ to obtain a black Hamilton cycle and join the red edges in the a_1^2 and a_4^2-columns. Choose x so that $(a_1, b_{1+x})(a_1, b_{2+x})$ is a red edge. Then apply an $\{a_2, b_{1+x}, b_{t_1+x}\}$-LAVS to obtain three monochromatic Hamilton cycles.

1.iii. $\boldsymbol{t_2 = 2k_2 + 1 > 2k_1 = t_1 \geqslant 4}$. Apply the CS-configuration of Lemma 2.2.9 to $H_1 \cup F$ to obtain a blue Hamilton cycle and join the red edges in the a_1 and a_4-columns. Choose x so that $(a_1^2, b_{1+x})(a_1^2, b_{2+x})$ is a red edge. Then apply an $\{a_2^2, b_{1+x}, b_{t_2+x}\}$-LAVS to obtain three monochromatic Hamilton cycles.

1.vi. $\boldsymbol{t_2 = 2k_2 + 1 \geqslant 2k_1 + 1 = t_1 \geqslant 3}$. First suppose that $m = t_2$. Hence, $|s_{j_2}| = 4$ and so $\langle s_{j_2} \rangle = \{0, \pm s_{j_2}, 2s_{j_2}\}$. By definition of Γ, $s_{j_1} \notin \langle s_{j_2} \rangle$, and as $m \geqslant 4$, and $m = 2k_2 + 1$, we have $m \geqslant 5$, so that $s_i \notin \langle s_{j_2} \rangle$. Therefore, as $\langle s_{j_2} \rangle$ is a subgroup of odd index, we are done by Lemma 5.3.1. Now suppose $m > t_2$. Hence, $m \geqslant 6$. Apply an $\{a_2, b_1, b_{t_1}\}$-LAVS to obtain a blue Hamilton cycle. By the justification of Theorem 2.2.7(a), all horizontal edges in the b_{t_1} and b_{t_1+1}-rows have the same direction upon making the blue cycle a directed cycle. Now, apply an $\{a_1, a_2, b_{t_1}, b_{t_1+1}\}$-CS to preserve the blue cycle. There now exists a red 2-factor consisting of three cycles, call them the ♠-cycle, the ♣-cycle, and the ♢-cycle. The ♠-cycle consists of the following:

$$(a_1, b_1)(a_2, b_1)(a_2, b_m)(a_2, b_{m-1}) \cdots (a_2, b_{t_1+1})(a_1, b_{t_1+2}) \cdots (a_1, b_m)(a_1, b_1)$$

The ♣-cycle consists of all vertical red edges in the a_3-column and all vertical red edges in the a_1-column that are between the b_1 and b_{t_1}-rows. The latter set forms a matching with the blue edges in the a_1-column. Finally, the ♢-cycle consists of all m vertical red edges in the a_4-column. Cyclically shift the columns in $H_2 \cup F$ so as to relocate the r_2-jump to be between the a_1^2 and a_2^2-columns. In this manner, we view $H_2 \cup F$ as $C_4^{(2)} \square_{r_2} C_m$ where $C_4^{(2)} := a_2^2 a_3^2 a_4^2 a_1^2 a_2^2$. Let x be the integer such that the set of vertical edges in the a_1^2-column between the b_{1+x} and b_{t_1+1+x}-rows is precisely the red-blue partial matching in that column, i.e., the edges $(a_1^2, b_{1+x})(a_1^2, b_{2+x})$ and $(a_1^2, b_{t_1+x})(a_1^2, b_{t_1+1+x})$ are both blue. As $m > t_2$, all vertical edges in the a_4^2 and a_1^2-columns between the b_{t_1+1+x} and b_{m+x}-rows are red. This forms a total of

$$(m + x) - (t_1 + 1 + x) + 1 = m - t_1 \geqslant m - t_2 \geqslant t_2$$

pairs of red edges. If $e := (a_3^2, b_{t_1+1+x})(a_3^2, b_{t_1+2+x})$ is red, apply an

$$\{a_4^2, b_{t_1+1+x}, b_{t_2+t_1+1+x}\}\text{-LAVS}$$

and if e is not red, apply the corresponding RAVS. The result joins the ♠, ♣, and ♢-cycles and produces three monochromatic Hamilton cycles.

Case 2: $\boldsymbol{n = 6}$**.** If $m = 3$, then $A \cong \mathbb{Z}_{18}$ or $A \cong \mathbb{Z}_3 \times \mathbb{Z}_6$. Thus, by the computational results of Table 1.2 in Appendix A, we may assume $m \geqslant 4$.

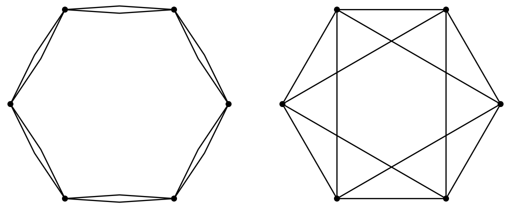


Figure 5.2: The quotient graphs of Case 2 of Lemma 5.3.3.

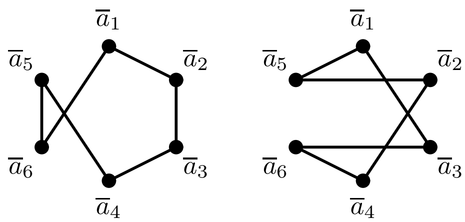


Figure 5.3: A Hamilton decomposition of $\Lambda_2 := \text{Cay}(\mathbb{Z}_6, \{1,2\})$ from Case 2 of Lemma 5.3.3.

Clearly, $A/J \cong \mathbb{Z}_6$, and up to relabeling, $\Delta = \Lambda_1 := \text{Cay}(\mathbb{Z}_6, \{\pm 1\})$ or $\Delta = \Lambda_2 := \text{Cay}(\mathbb{Z}_6, \{1,2\})$, both of which are depicted in Figure 5.2. If $\Delta = \Lambda_1$, then without loss of generality, $\pi_1 = \pi_2 = (1)$, and by Corollary 4.1.6, we may assume $t_1 \geqslant 3$. If $\Delta = \Lambda_2$, then using the Hamilton decomposition shown in Figure 5.3, we may assume $\pi_1 = (1)$ and $A_6^{(2)} := a_1^2 a_3^2 a_6^2 a_4^2 a_2^2 a_5^2 a_1^2$.

2.i. $\boldsymbol{t_2 = 2k_2 \geqslant 2k_1 = t_1 \geqslant 2}$. Here $m \geqslant 4$. If $m = 4$, then $|A| = 24$, hence,

$$A \cong \mathbb{Z}_{24},\ \mathbb{Z}_2 \times \mathbb{Z}_{12},\ \text{or } \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6.$$

By Theorem 1.5.2 or Tables 1.3, 1.4, and 1.5 of Appendix A, we may assume $m \geqslant 6$.

If $\Delta = \Lambda_1$, apply the CS-configuration of Lemma 2.2.9 to $H_1 \cup F$ to obtain a blue Hamilton cycle. Suppose a RAVS or LAVS were applied between the a_1^2 and a_3^2-columns, followed by the color-switches $\{\mathcal{X}_1, \mathcal{X}_2\}$, where $\mathcal{X}_1$ is c-incident to the a_3^2 and a_4^2-columns, and $\mathcal{X}_2$ is c-incident to the a_4^2 and a_5^2-columns. Clearly, a red Hamilton cycle would be formed. Making the red cycle a directed cycle, we see that all vertical red edges in the a_i^2-columns are $\uparrow$-edges, where $i = 1, 3, 5$, and all vertical red edges in the a_i^2-columns are $\downarrow$-edges, where $i = 2, 4$. By Lemma 2.2.9, the vertical red edges in the a_6^2-column are also $\uparrow$-edges. Hence, by Remark 2.2.2, applying the color-switch $\mathcal{X}_3$ that is c-incident to the a_5^2 and a_6^2-columns, will preserve the red Hamilton cycle. We shall now define $\{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3\}$. Choose x so that $(a_1^2, b_{1+x})(a_1^2, b_{2+x})$ is red, and let $e := (a_6^2, b_{t_2-1+x})(a_6^2, b_{t_2+x})$. If e is red, then apply an $\{a_2^2, b_{1+x}, b_{t_2-1+x}\}$-LAVS and define

$$\mathcal{X}_1 := \{a_3^2, a_4^2, b_{t_2-1+x}, b_{t_2+x}\}\text{-CS}.$$

We now have a black Hamilton cycle. By Lemma 2.2.15, define

$$\mathcal{X}_2 := \{a_4^2, a_5^2, b_{t_2-2+x}, b_{t_2-1+x}\}\text{-CS and } \mathcal{X}_3 := \{a_5^2, a_6^2, b_{t_2-1+x}, b_{t_2+x}\}\text{-CS}$$

to preserve the black and red Hamilton cycles. The result is a Hamilton decomposition of Γ. If e is non-red, i.e., e is blue, then clearly, $(a_6^2, b_{t_2-2+x})(a_6^2, b_{t_2-1+x})$ is red. Hence, apply an $\{a_2^2, b_{m+x}, b_{t_2-2+x}\}$-RAVS and define

$$\mathcal{X}_1 := \{a_3^2, a_4^2, b_{t_2-2+x}, b_{t_2-1+x}\}\text{-CS}$$

to obtain a black Hamilton cycle. By Lemma 2.2.15, define

$$\mathcal{X}_2 := \{a_4^2, a_5^2, b_{t_2-1+x}, b_{t_2+x}\}\text{-CS and } \mathcal{X}_3 := \{a_5^2, a_6^2, b_{t_2-2+x}, b_{t_2-1+x}\}\text{-CS}$$

to preserve the black and red Hamilton cycles. The result is a Hamilton decomposition of Γ.

If $\Delta = \Lambda_2$, and $t_2 = 2$, then apply an $\{a_1, a_2, b_1, b_2\}$-CS to obtain a blue Hamilton cycle. By Lemma 2.2.15, the application of a $\{a_2, a_6, b_2\}$-LAHS will preserve the blue cycle, and create a red Hamilton cycle. Clearly, the direction pattern of the red cycle is: $\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow$, in particular, all vertical red edges in the a_1 and a_3-columns have the same direction. As $m \geqslant 4$, there exists some integer x such that both $e = (a_1^2, b_{1+x})(a_1^2, b_{2+x})$ and $f = (a_3^2, b_{1+x})(a_3^2, b_{2+x})$ are red edges. Furthermore, e and f have the same direction, and the application of an $\{a_1^2, a_3^2, b_{1+x}, b_{2+x}\}$-CS yields the result. If $t_2 \geqslant 4$, then relocate the r_1-jump in $H_1 \cup F$ to between the a_5 and a_6-columns. In this manner, we view $H_1 \cup F$ as $A_6 \square_{r_1} B_m$, where $A_6 := a_6a_1a_2a_3a_4a_5a_6$. Apply the CS-configuration of Lemma 2.2.9 to $H_1 \cup F$ to obtain a blue Hamilton cycle and connect the vertical red edges in the a_5 and a_6-columns into one cycle. Choose x so that $(a_6^2, b_{1+x})(a_6^2, b_{2+x})$ is red, and let $e := (a_5^2, b_{t_2-1+x})(a_5^2, b_{t_2+x})$. We subdivide into two cases on the color of e.

(a) If e is red, then apply an $\{a_3^2, b_{1+x}, b_{t_2-1+x}\}$-RAVS and an

$$\{a_6^2, a_4^2, b_{t_2-1+x}, b_{t_2+x}\}\text{-CS}.$$

At this point, there exists a black Hamilton cycle. If we apply a color-switch c-incident to the a_4^2 and a_2^2-columns, we will obtain a red Hamilton cycle and break the black cycle. Making the red cycle a directed cycle, we see that, by Lemma 2.2.9, the vertical red edges in the a_6^2 and a_5^2-columns are $\uparrow$-edges. It follows that the vertical red edges in the a_4^2-column are $\downarrow$-edges, and thus the vertical red edges in the a_2^2-column are forced to be $\uparrow$-edges. By Remark 2.2.2, an additional color-switch c-incident to the a_2^2 and a_5^2-columns will preserve the red Hamilton cycle. Hence, apply an

$$\{a_4^2, a_2^2, b_{t_2-2+x}, b_{t_2-1+x}\}\text{-CS and an } \{a_2^2, a_5^2, b_{t_2-1+x}, b_{t_2+x}\}\text{-CS}.$$

By Lemma 2.2.15, the result is three monochromatic Hamilton cycles.

(b) If e is non-red, i.e., e is blue, then relocate the r_2-jump to be between the a_6^2 and a_3^2-columns. In this manner, we view $H_2 \cup F = A_6^{(2)} \square_{r_2} B_m$ where

$$A_6^{(2)} := a_6^2a_4^2a_2^2a_5^2a_1^2a_3^2a_6^2.$$

Apply an $\{a_6^2, a_4^2, b_{1+x}, b_{2+x}\}$-CS and an $\{a_4^2, a_2^2, b_{2+x}, b_{3+x}\}$-CS to connect all vertical red edges in the a_i^2-columns, where $i \in \{6, 4, 2, 5\}$, into one cycle. Apply an $\{a_1^2, b_{3+x}, b_{t_2-1+x}\}$-RAVS to create a red Hamilton cycle. As before, making the red cycle a directed cycle, it is clear the the vertical red edges in the a_i^2-columns take the form: $\uparrow\downarrow\uparrow\uparrow\downarrow\uparrow$. Hence, we may apply an additional switch, $\mathcal{X}$, c-incident to the a_3^2 and a_6^2-columns, and preserve the red cycle, by Remark 2.2.2. Consider the following 4-cycle:

$$(a_3^2, b_{t_2-1+x}) \xrightarrow{e_1} (a_6^2, b_{t_2-1+r_2+x}) \xrightarrow{e_2} (a_6^2, b_{t_2+r_2+x}) \xrightarrow{e_3} (a_3^2, b_{t_2+x}) \xrightarrow{e_4} (a_3^2, b_{t_2-1+x}).$$

Clearly, e_4 is red, and e_1 and e_3 are black. By our choice of x, and the fact that r_2 is even, it can never be the case that e_2 is blue. If e_2 is red, then let

$$\mathcal{X} := \{a_3^2, a_6^2, b_{t_2-1+x}, b_{t_2+x}\}\text{-CS}$$

to obtain a black Hamilton cycle. If e_2 is non-red, then

$$(a_6^2, b_{t_2-1+r_2+x}) = (a_6^2, b_{1+x}),$$

and hence, it is the only black edge in the a_6^2-column. In this case, remove the

$$\{a_1^2, b_{3+x}, b_{t_2-1+x}\}\text{-RAVS},$$

and apply an $\{a_1^2, b_{4+x}, b_{t_2+x}\}$-LAVS. Define $\mathcal{X} := \{a_3^2, a_6^2, b_{3+x}, b_{4+x}\}$-CS to obtain a Hamilton decomposition of Γ.

2.ii. $\boldsymbol{t_2 = 2k_2 > 2k_1 + 1 = t_1 \geqslant 3}$. Note $m \geqslant 6$.

If $\Delta = \Lambda_1$, then apply the CS-configuration of Lemma 2.2.15 to $H_2 \cup F$ and let x be the integer such that $(a_1, b_{1+x})(a_2, b_{2+x})$ is red. Apply an $\{a_2, b_{1+x}, b_{t_1+x}\}$-LAVS and an $\{a_3, a_5, b_{t_1+x}\}$-LAHS to obtain the result.

If $\Delta = \Lambda_2$, then apply an $\{a_2, b_1, b_{t_1}\}$-LAVS to obtain a blue Hamilton cycle. There exists a path of $t_1 - 1 < m$ vertical blue edges in the a_2-column. Let x be any integer such that $(a_2^2, b_{t_2-1+x})(a_2^2, b_{t_2+x})$ is red and apply an $\{a_2^2, a_5^2, b_{t_2-1+x}, b_{t_2+x}\}$-CS. Finally, apply an

$$\begin{cases} \{a_6^2, b_{1+x}, b_{t_2-1+x}\}\text{-LAVS} & \text{if } (a_3^2, b_{1+x})(a_3^2, b_{2+x}) \text{ is red} \\ \{a_6^2, b_{1+x}, b_{t_2-1+x}\}\text{-RAVS} & \text{if } (a_3^2, b_{1+x})(a_3^2, b_{2+x}) \text{ is not red.} \end{cases}$$

2.iii. $\boldsymbol{t_2 = 2k_2 + 1 > 2k_1 = t_1 \geqslant 2}$. Here $m \geqslant 2t_2 \geqslant 6$.

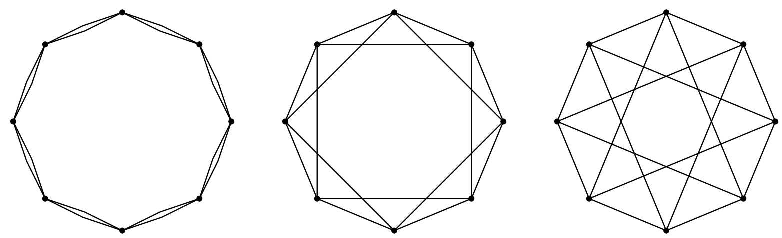
If $\Delta = \Lambda_1$, then use the method of Case 2.ii by applying the CS-configuration of Lemma 2.2.15 to $H_1 \cup F$.

If $\Delta = \Lambda_2$, and $t_1 = 2$, apply an $\{a_4^2, b_{1+x}, b_{t_2+x}\}$-LAVS to $H_2 \cup F$ to obtain a black Hamilton cycle, where x is chosen so that $(a_4, b_1)(a_4, b_2)(a_4, b_3)$ is a red 3-path. Apply the following three switches to $H_1 \cup F$ to obtain a Hamilton decomposition:

$$\begin{cases} \{a_1, a_3, b_2\}\text{-RAHS and an } \{a_4, a_5, b_1, b_2\}\text{-CS} & \text{if } (a_3, b_2)(a_3, b_3) \text{ is red} \\ \{a_1, a_3, b_2\}\text{-LAHS and an } \{a_4, a_5, b_2, b_3\}\text{-CS} & \text{if } (a_3, b_2)(a_3, b_3) \text{ is not red.} \end{cases}$$

If $t_1 > 2$, then apply an $\{a_4^2, b_{1+x}, b_{t_2+x}\}$-LAVS to $H_2 \cup F$, where this time, x is chosen so that $(a_4, b_{t_1-1})(a_4, b_{t_1})$ is red and apply an $\{a_4, a_5, b_{t_1-1}, b_{t_1}\}$-CS. Finally, apply an

$$\begin{cases} \{a_2, b_1, b_{t_1-1}\}\text{-RAVS} & \text{if } (a_3, b_1)(a_3, b_2) \text{ is red} \\ \{a_2, b_1, b_{t_1-1}\}\text{-LAVS} & \text{if } (a_3, b_1)(a_3, b_2) \text{ is not red.} \end{cases}$$



Λ_1 Λ_2 Λ_3

Figure 5.4: The quotient graphs of Case 3 of Lemma 5.3.3.

2.vi. $\boldsymbol{t_2 = 2k_2 + 1 \geqslant 2k_1 + 1 = t_1 \geqslant 3}$. Here $m \geqslant 3$.

If $\Delta = \Lambda_1$, then apply an $\{a_2, b_1, b_{t_1}\}$-LAVS and choose x so that $(a_3^2, b_{1+x})(a_3^2, b_{2+x})$ is red. Apply an $\{a_4^2, b_{1+x}, b_{t_2+x}\}$-LAVS and an $\{a_5^2, a_6^2, b_{t_2+x}, b_{t_2+1+x}\}$-CS to obtain the result.

If $\Delta = \Lambda_2$, then apply an $\{a_2, b_1, b_{t_1}\}$-LAVS to obtain a blue Hamilton cycle. There exists a path of $t_1 - 1 < m$ vertical blue edges in the a_2-column. Let x be any integer such that $(a_2^2, b_{t_2+x})(a_2^2, b_{t_2+1+x})$ is red and apply an $\{a_2^2, a_5^2, b_{t_2+x}, b_{t_2+1+x}\}$-CS. Finally, apply an

$$\begin{cases} \{a_6^2, b_{1+x}, b_{t_2+x}\}\text{-LAVS} & \text{if } (a_3^2, b_{1+x})(a_3^2, b_{2+x}) \text{ is red} \\ \{a_6^2, b_{1+x}, b_{t_2+x}\}\text{-RAVS} & \text{if } (a_3^2, b_{1+x})(a_3^2, b_{2+x}) \text{ is not red.} \end{cases}$$

By Theorem 2.2.7(a), the result is a Hamilton decomposition.

Case 3: $\boldsymbol{n = 8}$**.** As before, we must have $|\overline{s_{j_i}}| \in \{4, 8\}$. If $|\overline{s_{j_1}}| = 8$, then $A/J \cong \mathbb{Z}_8$, and, up to a relabeling of the vertices, Δ is either the graph $\Lambda_1 := \textsc{Cay}(\mathbb{Z}_8, \{\pm 1\})$, so without loss of generality, $\pi_1 = \pi_2 = (1)$, or $\Lambda_2 := \textsc{Cay}(\mathbb{Z}_8, \{1, 2\})$, so without loss of generality, $\pi_1 = (1)$ and $\pi_2 = (2356487)$. In this case,

$$A_8^{(2)} := a_1^2 a_3^2 a_5^2 a_8^2 a_6^2 a_4^2 a_2^2 a_7^2 a_1^2.$$

If $|\overline{s_{j_1}}| = |\overline{s_{j_2}}| = 4$, then $A/J \cong \mathbb{Z}_4 \times \mathbb{Z}_2$, and, up to a relabeling of the vertices, Δ is the graph $\Lambda_3 := \textsc{Cay}(\mathbb{Z}_4 \times \mathbb{Z}_2, \{(1,0), (1,1)\})$, so without loss of generality, $\pi_1 = (1)$ and $\pi_2 = (24)(37)(68)$. In this case,

$$A_8^{(2)} := a_1^2 a_4^2 a_7^2 a_2^2 a_5^2 a_8^2 a_3^2 a_6^2 a_1^2.$$

In what follows, we shall write $\Delta = \Lambda_i$ to mean, Δ and Λ_i are equal as unlabeled graphs, and these are shown in Figure 5.4. As before, if $\Delta \in \{\Lambda_1, \Lambda_3\}$, then by Corollary 4.1.6, we may assume $t_1 \geqslant 3$. Cases 3.ii-3.iv for $\Delta = \Lambda_1$, follow from the technique of Cases 2.ii-2.iv., respectively. If $m = 3$, then $|A| = 24$, which is resolved in Tables 1.3, 1.4, and 1.5 of Appendix A. If $m = 4$, then $|A| = 32$, and so A is isomorphic to one of four possible abelian groups. Note, $A \ncong \mathbb{Z}_2^5$, for we are requiring that $|s_i| \geqslant 3$ for all $1 \leqslant i \leqslant 3$, and $A \ncong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$, because $|S| = 3$. Thus,

$$A \cong \mathbb{Z}_{32},\ \mathbb{Z}_4 \times \mathbb{Z}_8,\ \mathbb{Z}_2 \times \mathbb{Z}_{16},\ \text{or } \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_8.$$

These small cases are resolved in Tables 1.6, 1.7, and 1.8 of Appendix A. Thus, we assume $m \geqslant 5$.

3.i. $\boldsymbol{t_2 = 2k_2 \geqslant 2k_1 = t_1 \geqslant 4}$.

If $\Delta = \Lambda_1$, then apply the CS-configuration of Lemma 2.2.9 to $H_1 \cup F$ and choose x so that $(a_1^2, b_{1+x})(a_2^2, b_{2+x})$ is red. Let $e := (a_8^2, b_{b_{t_2-1+x}})(a_8^2, b_{t_2+x})$. If e is not red, then apply an $\{a_1^2, a_2^2, b_{1+x}, b_{2+x}\}$-CS, and an $\{a_2^2, a_4^2, b_{t_2-1+x}\}$-LAHS. We now have obtained a blue Hamilton cycle, a black Hamilton cycle, and a red 2-factor consisting of one cycle on the vertical red edges in the a_i^2-columns, where $i \in \{1, 2, 3, 4, 8\}$. As $(a_8^2, b_{t_2-2+x})(a_8^2, b_{t_2-1+x})$ is red, we may apply an $\{a_4^2, a_8^2, b_{t_2-1+x}\}$-LAHS to preserve the black cycle, by Lemma 2.2.15. To see that this produces a red Hamilton cycle, we note that prior to applying the $\{a_7^2, a_8^2, b_{t_2-1+x}, b_{t_2+x}\}$-CS, a red Hamilton cycle exists. As the red edges in the a_1^2 and a_8^2-columns are $\uparrow$-edges, the direction pattern is:

$$\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\uparrow.$$

Hence, the $\{a_7^2, a_8^2, b_{t_2-1+x}, b_{t_2+x}\}$-CS preserves the red Hamilton cycle. If e is red, then apply an $\{a_2^2, b_{1+x}, b_{t_2-1+x}\}$-LAVS, and an $\{a_3^2, a_4^2, b_{t_2-1+x}, b_{t_2+x}\}$-CS. The application of an $\{a_4^2, a_8^2, b_{t_2-1+x}\}$-RAHS creates a set of three monochromatic Hamilton cycles.

If $\Delta = \Lambda_2$, and $t_2 = 2$, follow the technique of Case 2.i. Relocate the r_1-jump in $H_1 \cup F$ to between the a_2 and a_3-columns. In this manner, we view $H_1 \cup F$ as $A_8 \,\square_{r_1}\, B_m$, where

$$A_8 := a_3 a_4 a_5 a_6 a_7 a_8 a_1 a_2 a_3.$$

Apply the CS-configuration of Lemma 2.2.9 to $H_1 \cup F$ to obtain a blue Hamilton cycle and connect the vertical red edges in the a_2 and a_3-columns into one cycle. Choose x so that $(a_3^2, b_{1+x})(a_3^2, b_{2+x})$ is red, and let $e := (a_2^2, b_{t_2-1+x})(a_2^2, b_{t_2+x})$. We subdivide into two cases on the color of e.

(a) If e is red, then apply an $\{a_1^2, b_{1+x}, b_{t_2-1+x}\}$-RAVS and an $\{a_3^2, a_5^2, b_{t_2-1+x}, b_{t_2+x}\}$-CS to obtain a black Hamilton cycle. Next, apply an $\{a_5^2, a_6^2, b_{t_2-1+x}\}$-RAHS, which by Lemma 2.2.15, preserves the black cycle. As before, if an

$$\{a_6^2, a_4^2, b_{t_2-2+x}, b_{t_2-1+x}\}\text{-CS}$$

is applied, a red Hamilton cycle is created. Upon making this a directed cycle, the column direction pattern of

$$A_8^{(2)} := a_7^2 a_1^2 a_3^2 a_5^2 a_8^2 a_6^2 a_4^2 a_2^2$$

is

$$\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\uparrow.$$

Thus, applying an $\{a_4^2, a_2^2, b_{t_2-1+x}, b_{t_2+x}\}$-CS, will preserve the red cycle, and by Lemma 2.2.15, preserve the black cycle. The result is now obtained.

(b) If e is non-red, i.e., e is blue, then relocate the r_2-jump to be between the a_3^2 and a_1^2-columns. In this manner, we view $H_2 \cup F = A_6^{(2)} \,\square_{r_2}\, B_m$ where

$$A_6^{(2)} := a_3^2 a_5^2 a_8^2 a_6^2 a_4^2 a_2^2 a_7^2 a_1^2 a_3^2.$$

Apply an $\{a_3^2, a_4^2, b_{2+x}\}$-RAHS to connect all horizontal black edges in the b_{j+x}-rows, where $j \in \{1, 2, 3\}$, to one cycle, and connect all vertical red edges in the a_i^2-columns, where $i \in \{3, 5, 8, 6, 4, 2\}$, into one cycle. The result now follows by finding three additional switches, $\{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3\}$, between the a_2^2, a_7^2, a_1^2, and a_3^2-columns, by using the technique of Case 2.i.(b).

If $\Delta = \Lambda_3$, then apply the CS-configuration of Lemma 2.2.9 to $H_2 \cup F = A_8^{(2)} \square_{r_2} B_m$ to obtain a black Hamilton cycle and join the vertical red edges in the a_1^2 and a_6^2-columns into one cycle. We can assume the switch was applied so that $(a_1, b_1)(a_1, b_2)$ is a red edge in $A_8^{(1)} \square_{r_2} B_m$. Let $e := (a_6, b_{t_1-1})(a_6, b_{t_1})$, and let v be the vertex $(a_1, b_{t_1-1+r_1})$. Note, $v \neq (a_1, b_1)$, for otherwise, the blue edges in the b_1 and b_{t_1-1}-rows must have been on the same cycle, a contradiction.

(a) If e is red, then apply an

$$\{a_2, b_1, b_{t_1-1}\}\text{-LAVS and an } \{a_3, a_4, b_{t_1-1}, b_{t_1}\}\text{-CS}.$$

This produces a blue Hamilton cycle, and connects the a_i-columns, $i \in \{1, 2, 3, 4, 6\}$ into a single red cycle. As e is red, apply an $\{a_4, a_6, b_{t_1-1}\}$-RAHS, an $\{a_7, a_8, b_{t_1-2}, b_{t_1-1}\}$-CS, and an $\{a_8, a_1, b_{t_1-1}, b_{t_1}\}$-CS (because r_1 is even, the last switch is well-defined). We claim this provides a Hamilton decomposition of Γ. To see this, note that if the $\{a_5, a_6, b_{t_1-1}, b_{t_1}\}$-CS is removed, we have a red Hamilton cycle with column-direction pattern is:

$$\uparrow\downarrow\uparrow\downarrow\uparrow\uparrow\uparrow\downarrow.$$

Hence, applying an $\{a_5, a_6, b_{t_1-1}, b_{t_1}\}$-CS will preserve the red cycle by Remark 2.2.2, and by Lemma 2.2.15, the blue Hamilton cycle is preserved.

(b) If e is not red, and $t_1 \geqslant 6$, then apply an $\{a_1, a_5, b_2\}$-RAHS, an $\{a_7, b_3, b_{t_1-1}\}$-RAVS, and an $\{a_8, a_1, b_{t_1-1}, b_{t_1}\}$-CS to obtain the result. Use a similar technique for the case $t_1 = 4$.

3.ii. $\boldsymbol{t_2 = 2k_2 > 2k_1 + 1 = t_1 \geqslant 3}$.

If $\Delta = \Lambda_2$, then apply an $\{a_2, b_1, b_{t_1}\}$-LAVS to obtain a blue Hamilton cycle. Note that as $m \geqslant t_2 > t_1$, there exists a red 3-path P in the a_2-column. Let x be the integer such that P lies in between the b_{t_2-2+x} and b_{t_2+x}-rows. Apply an $\{a_8^2, b_{1+x}, b_{t_2-1+x}\}$-LAVS and an

$$\{a_6^2, a_4^2, b_{t_2-1+x}, b_{t_2+x}\}\text{-CS}$$

to create a black Hamilton cycle. By Lemma 2.2.15, applying an $\{a_4^2, a_7^2, b_{t_2-1+x}\}$-RAHS will preserve the black cycle, and create a Hamilton decomposition of Γ.

If $\Delta = \Lambda_3$, let x be the integer such that $e := (a_6, b_m)(a_6, b_1)$ in $H_1 \cup F$ corresponds to $(a_6^2, b_{t_2-1+x})(a_6^2, b_{t_2+x})$ in $H_2 \cup F$. Apply an

$$\{a_4^2, b_{1+x}, b_{t_2-1+x}\}\text{-RAVS and an } \{a_7^2, a_2^2, b_{t_2-1+x}, b_{t_2+x}\}\text{-CS}$$

to obtain a black Hamilton cycle and join all vertical red edges in the a_i^2-columns, where $i \in \{1, 4, 7, 2\}$, into one cycle C. Apply an $\{a_8^2, a_6^2, b_{t_2-1+x}\}$-RAHS to preserve the black cycle, by Lemma 2.2.15, and create a red cycle, C', on the a_i^2-columns, where $i \in \{8, 3, 6\}$. Note, that e is now the only non-red edge in the a_6-column, and the a_7-column contains a t_2-path of alternating red and black edges. Therefore, apply an $\{a_6, b_1, b_{t_1}\}$-LAVS or -RAVS, depending on which defines a valid switching configuration, to obtain a Hamilton decomposition of Γ.

3.iii. $\boldsymbol{t_2 = 2k_2 + 1 > 2k_1 = t_1 \geqslant 4}$.

If $\Delta = \Lambda_2$, then apply an $\{a_2, b_1, b_{t_1-1}\}$-LAVS and an $\{a_6, a_7, b_{t_1-1}, b_{t_1}\}$-CS. We now have a blue Hamilton cycle and a red 2-factor consisting, among other things, of a red cycle, C, on the a_i-columns, where $i \in \{1, 2, 3\}$, and a red cycle, C', on the a_6 and a_7-columns. Let x be an integer such that the $(a_6^2, b_{1+x})(a_6^2, b_{2+x})$ is a blue edge (note this is the only non-red edge in the a_6^2-column). Apply an $\{a_8^2, b_{1+x}, b_{t_2+x}\}$-LAVS to create a black Hamilton cycle and join the vertical red edges in the a_5^2 and a_8^2-columns to C'. Call this new cycle, C''. The application of an $\{a_6^2, a_2^2, b_{t_2+x}\}$-LAHS will join C and C'' into a red Hamilton cycle. By Lemma 2.2.15, this also will preserve the black Hamilton cycle. The result now follows.

If $\Delta = \Lambda_3$, then $t_1 \geqslant 4$, and $t_2 \geqslant 5$. Apply the color switching configuration of Lemma 2.2.9 to $H_1 \cup F$. Let x be the integer such that $(a_1^2, b_{1+x})(a_1^2, b_{2+x})$ is red. Apply an $\{a_4^2, b_{1+x}, b_{t_2+x}\}$-LAVS to obtain a black Hamilton cycle. If $h := (a_8^2, b_{t_2+x})(a_8^2, b_{t_2+1+x})$ is red, then apply an $\{a_7^2, a_5^2, b_{t_2+x}\}$-LAHS and an $\{a_8^2, a_6^2, b_{t_2+x}\}$-LAHS to obtain the result. If h is not red, then clearly $(a_8^2, b_{2j+x})(a_8^2, b_{2j+1+x})$ is red, for all j. Apply an $\{a_4^2, a_8^2, b_{2+x}\}$-RAHS to connect all horizontal black edges in the b_{1+x}, b_{2+x}, and b_{3+x}-rows into a single cycle. Next, apply an $\{a_3^2, b_{3+x}, b_{t_2+x}\}$-RAVS to create a black Hamilton cycle, and obtain the result.

3.vi. $\boldsymbol{t_2 = 2k_2 + 1 \geqslant 2k_1 + 1 = t_1 \geqslant 3}$.

If $\Delta = \Lambda_2$, then apply an $\{a_2, b_1, b_{t_1}\}$-LAVS and an $\{a_3, a_5, b_{t_1}\}$-LAHS, which by Lemma 2.2.15 produces a blue Hamilton cycle, and joins the vertical red edges of the a_i-columns, where $1 \leqslant i \leqslant 5$, into one cycle, C. There exists at least one red edge in the a_2^2-column. Let x be the integer such that $(a_2^2, b_{t_2+x})(a_2^2, b_{t_2+1+x})$ is red. Furthermore, note that

$$h := (a_5, b_{t_1-1})(a_5, b_{t_1}) = (a_5^2, b_y)(a_5^2, b_{y+1})$$

is the only non-red edge in the a_5^2-column. If y is odd, then apply an $\{a_8^2, b_{1+x}, b_{t_2+x}\}$-RAVS, and if y is even, then apply an $\{a_8^2, b_{1+x}, b_{t_2+x}\}$-LAVS to obtain a black Hamilton cycle. Finally, by Corollary 2.2.8(a), the application of an $\{a_2^2, a_7^2, b_{t_2+x}, b_{t_2+1+x}\}$-CS preserves the black cycle, and creates a red Hamilton cycle, to obtain a Hamilton decomposition of Γ.

If $\Delta = \Lambda_3$, then apply an $\{a_4^2, b_{1+x}, b_{t_2+x}\}$-LAVS, where x is chosen so that

$$e := (a_4, b_{t_1})(a_4, b_{t_1+1}) = (a_4^2, b_{t_2+x})(a_4^2, b_{t_2+1+x}).$$

The choice of x guarantees that e is red. Let $f := (a_7, b_{t_1})(a_7, b_{t_1+1})$. If f is red, then apply an $\{a_4, a_6, b_{t_1}\}$-LAHS and an $\{a_7, a_8, b_{t_1}, b_{t_1+1}\}$-CS. Apply either an $\{a_2, b_1, b_{t_1}\}$-LAVS or -RAVS, depending on which defines a good red-blue CS-configuration. In either case, by Corollary 2.2.8(a) and Lemma 2.2.15, the result is a Hamilton decomposition. If f is not red, then switch reflect the $\{a_4^2, b_{1+x}, b_{t_2+x}\}$-LAVS so it becomes a -RAVS. Now, clearly, f is red, and e is still red. Use the previous technique to obtain a Hamilton decomposition of Γ. ■

5.4 Main Results

If $\Delta_i = \textsc{Cay}(A/\langle s_i\rangle, \{\overline{s_{j_1}}, \overline{s_{j_2}}\})$ is a connected, 4-regular, quotient graph of $\textsc{Cay}(A, \{s_1, s_2, s_3\})$, it cannot have order $n = 2$, for each connection set element generates a 1-factor. Furthermore, if $|V(\Delta_i)| = 3$, then $\Delta_i \simeq \textsc{Cay}(\mathbb{Z}_3, \{1, 1\})$, which is a multigraph. Thus, Theorem 5.2.9 and Lemmata 5.3.1 and 5.3.3 combine to yield the following result.

Theorem 5.4.1. *If $\Gamma = \textsc{Cay}(A, \{s_1, s_2, s_3\})$ is a connected, 6-regular, abelian Cayley graph of even order, and for some $1 \leqslant i \leqslant 3$, $\Delta_i = \textsc{Cay}(A/\langle s_i\rangle, \{\overline{s_{j_1}}, \overline{s_{j_2}}\})$ is 4-regular, and $\Delta_i \not\simeq \textsc{Cay}(\mathbb{Z}_3, \{1, 1\})$, then Γ has a Hamilton decomposition.*

Alternatively, Theorem 5.4.1 may be stated as follows.

Theorem 5.4.2. *If $\Gamma = \textsc{Cay}(A, S)$ is a connected, 6-regular, abelian Cayley graph of even order, then Γ has a Hamilton decomposition if S has no involutions, and for some $s \in S$, $\textsc{Cay}(A/\langle s\rangle, \overline{S})$ is 4-regular, and of order at least 4.*

Corollary 5.4.3. *If $\Gamma = \textsc{Cay}(A, \{s_1, s_2, s_3\})$ is a connected, 6-regular, abelian Cayley graph of even order, then Γ has a Hamilton decomposition if one of the following holds:*

(a) $s_1 \in \langle s_2, s_3\rangle$, $s_2 \in \langle s_1, s_3\rangle$, *and* $[A : \langle s_3\rangle] \geqslant 4$, *or*

(b) $|s_1| \geqslant |s_2| > 2|s_3|$, *or*

(c) $|s_1| \geqslant |s_2| > |s_3|$, *and either*

  i. $|A| = (2k+1)|s_3|$, *with* $k \geqslant 2$, *or*

  ii. $|A| \geqslant 4|s_3|$ *and* $|s_1|$ *and* $|s_2|$ *are odd.*

Proof. Clearly, $|s_i| \geqslant 3$ for $1 \leqslant i \leqslant 3$. Suppose (a), $\langle \overline{s_1}\rangle = \langle \overline{s_2}\rangle = A/\langle s_3\rangle$, and the result follows from Theorem 5.4.1. Suppose (b) holds, and note that if $|s_i|$ is even, then $|2s_i| = |s_i|/2$, and if $|s_i|$ is odd, then $|2s_i| = |s_i|$. Therefore, by Lagrange's Theorem, $2s_i \notin \langle s_3\rangle$, for $i = 1, 2$ and by Theorem 1.5.2, it is assumed $|A| \geqslant 2|s_1| > 4|s_3|$, so that $|A : \langle s_3\rangle| > 4$. Hence, the quotient graph $\textsc{Cay}(A/\langle s_3\rangle, \{\overline{s_1}, \overline{s_2}\})$ satisfies Theorem 5.4.1. Now suppose (c) holds. If s_3 generates a subgroup of odd index, at least five, clearly $\textsc{Cay}(A/\langle s_3\rangle, \{\overline{s_1}, \overline{s_2}\})$ is 4-regular, and we are done by Theorem 5.4.1. A similar result is obtained when $|s_1|$ and $|s_2|$ are odd, and s_3 generates a subgroup of index at least four. ■

Chapter 6

Conclusions and Further Research Problems

6.1 Open cases

One glaring omission to Theorem 5.4.1 is the following:

Open Problem 1. *If* $\Gamma = \text{Cay}(A, \{s_1, s_2, s_3\})$ *is a connected, 6-regular, abelian Cayley graph of even order, and for some* $1 \leqslant i \leqslant 3$*, the graph* $\text{Cay}(A/\langle s_i \rangle, \{\overline{s_{j_1}}, \overline{s_{j_2}}\}) \simeq \text{Cay}(\mathbb{Z}_3, \{1, 1\})$*, prove that* Γ *has a Hamilton decomposition.*

We give a partial solution to the above problem, in the case when s_{j_1} and s_{j_2} generate subgroups of even index in A.

Lemma 6.1.1. *If* $\Gamma = \text{Cay}(A, \{s_1, s_2, s_3\})$ *is a connected, 6-regular, abelian Cayley graph of even order, and for some* $1 \leqslant i \leqslant 3$*, the graph* $\text{Cay}(A/\langle s_i \rangle, \{\overline{s_{j_1}}, \overline{s_{j_2}}\}) \simeq \text{Cay}(\mathbb{Z}_3, \{1, 1\})$*, where* s_{j_1} *and* s_{j_2} *generate subgroups of even index in* A*, then* Γ *has a Hamilton decomposition.*

Proof. Let $m = |s_i|$, and without loss of generality, $|s_{j_1}| \geqslant |s_{j_2}|$. By Theorem 3.1.6, Γ is a $D(3, m, 3)$-graph, where $H_1 \cup F = A_3^{(1)} \square_{r_1} B_m$ consists of $t_1 = \gcd(r_1, m) = [A : \langle s_{j_1} \rangle] = 2k_1$ horizontal blue cycles. Similarly, $H_2 \cup F = A_3^{(2)} \square_{r_2} B_m$ consists of $t_2 = \gcd(r_2, m) = [A : \langle s_{j_2} \rangle] = 2k_2$ horizontal black cycles. We may assume that

$$A_3^{(1)} = a_1 a_2 a_3 a_1 \text{ and } A_3^{(2)} = a_1^2 a_2^2 a_3^2 a_1^2.$$

The case $m \in \{4, 6, 8\}$ is resolved in Appendix A. Hence, assume $m \geqslant 10$ is even, and by Corollary

4.1.6, assume $t_2 \geqslant t_1 \geqslant 4$. Apply the CS-configuration of Lemma 2.2.9 to $H_1 \cup F$ to obtain a blue Hamilton cycle and join the a_1 and a_3-columns into one red cycle. Relocate the r_2-jump in $H_2 \cup F$ so that it is between the a_2^2 and a_3^2-columns. In this manner, we view $H_2 \cup F$ as $A_3^{(2)} \square_{r_2} B_m$ with $A_3^{(2)} := a_3^2\, a_1^2\, a_2^2\, a_3^2$. Let h be any integer such that $(a_3^2, b_{1+h})(a_3^2, b_{2+h})$ is red. Now, apply the CS-configuration of Lemma 2.2.9 to $H_2 \cup F$ to create a black Hamilton cycle, and obtain a Hamilton decomposition of Γ. ■

6.1.1 Quotient connection sets with involutions

We briefly examine the case where, for all $i \in \{1,2,3\}$, the quotient graph $\mathrm{Cay}(A/\langle s_i\rangle, \{\overline{s_{j_1}}, \overline{s_{j_2}}\})$ of $\mathrm{Cay}(A, \{s_1, s_2, s_3\})$, is k-regular, for some $k \leqslant 3$. Note, $k = 0$ if and only if $\{s_{j_1}, s_{j_2}\} \subseteq \langle s_i\rangle$, i.e. A is a cyclic group generated by s_i. By Theorem 1.5.2, Γ is Hamilton decomposable. Thus, we assume $k \in \{1,2,3\}$. Equivalently, for all $i \in \{1,2,3\}$, the subgroup $\langle s_i\rangle$ contains $2s_j$, for some $j \neq i$, i.e., $\overline{S}$ contains at least one involution.

Suppose that $\Delta = \mathrm{Cay}(A/\langle s_i\rangle, \{\overline{s_{j_1}}, \overline{s_{j_2}}\})$ is 3-regular, of order n. It is clear that, without loss of generality, $|\overline{s_{j_2}}| = 2$, and thus, n is even, and

$$|A/\langle s_i\rangle| = n = \frac{|\langle\overline{s_{j_1}}\rangle||\langle\overline{s_{j_2}}\rangle|}{|\langle\overline{s_{j_1}}\rangle \cap \langle\overline{s_{j_2}}\rangle|}.$$

Hence, $|\overline{s_{j_1}}| \in \{n/2, n\}$, where $|\overline{s_{j_1}}| = n$ if $\overline{s_{j_2}} \in \langle\overline{s_{j_1}}\rangle$, and $|\overline{s_{j_1}}| = n/2$ if the intersection is trivial. If $|\overline{s_{j_1}}| = n \Rightarrow A/\langle s_i\rangle = \langle\overline{s_{j_1}}\rangle$, and so $A = \langle s_i, s_{j_1}\rangle$.

If $\{\overline{x}, \overline{y}\} \in E(\Delta)$ is generated by $\overline{s_{j_2}}$, then as $\overline{x} - \overline{y} = \overline{s_{j_2}} = -\overline{s_{j_2}} = \overline{-s_{j_2}}$, we have

$$(x + J) - (y + J) = s_{j_2} + J = -s_{j_2} + J,$$

so that

$$x + h_1 s_i - (y + k_1 s_i) = s_{j_2} \text{ and } x + h_2 s_i - (y + k_2 s_i) = -s_{j_2}.$$

Thus, as $s_{j_2} \neq -s_{j_2}$, $\{x, y + (k_1 - h_1)s_i\} = \{x, x - s_{j_2}\}$ and $\{x, y + (k_2 - h_2)s_i\} = \{x, x + s_{j_2}\}$ are distinct edges in Γ. Thus, the multi-edge $\{\overline{x}, \overline{y}\}$ lifts to:

$$L_\Delta\{\overline{x}, \overline{y}\} = \{\{x + ts_i - s_{j_2}, x + ts_i\} : t = 0, \ldots, m-1\} \cup \{\{x + ts_i, x + ts_i + s_{j_2}\} : t = 0, \ldots, m-1\}$$

As $2s_{j_2} \in \langle s_i\rangle$, we have $2s_{j_2} = ds_i$ for some $d > 0$. Thus, the union of the edges corresponding to values of $t \in \{0, d, 2d, \ldots\}$ is just the cycle containing x, that is generated by s_{j_2} in Γ.

Open Problem 2: *If $\Gamma = \mathrm{Cay}(A, \{s_1, s_2, s_3\})$ is a connected, 6-regular, abelian Cayley graph on A, such that for all $1 \leqslant i \leqslant 3$, the quotient graph $\mathrm{Cay}(A/\langle s_i\rangle, \{\overline{s_{j_1}}, \overline{s_{j_2}}\})$ has a connection set containing at least one involution, then show Γ has a Hamilton decomposition.*

6.1.2 General connection sets with involutions

Any connected, 6-regular abelian Cayley graph of A with connection set S must take the form:

1. $\textsc{Cay}(A, \{s_1, s_2, s_3\})$ with $|s_1| \geqslant |s_2| \geqslant |s_3| \geqslant 3$.
2. $\textsc{Cay}(A, \{s_1, s_2, s_3, s_4\})$ with $|s_3| = |s_4| = 2$ and $|s_1| \geqslant |s_2| \geqslant 3$.
3. $\textsc{Cay}(A, \{s_1, s_2, s_3, s_4, s_5\})$ with $|s_2| = |s_3| = |s_4| = |s_5| = 2$ and $|s_1| \geqslant 3$.
4. $\textsc{Cay}(A, \{s_1, s_2, s_3, s_4, s_5, s_6\})$ with $|s_i| = 2$ for all $1 \leqslant i \leqslant 6$.

A solution to Open Problems 1 and 2, together with Theorem 5.4.2, would prove Alspach's conjecture for Cayley graphs of type (1) above.

Open Problem 3: *If $\Gamma = \textsc{Cay}(A, S)$ is a connected, 6-regular, abelian Cayley graph on A, find a Hamilton decomposition of Γ if S contains involutions.*

Lemma 6.1.2. (Liu [32]) *Let $S = \{s_1, s_2, \ldots, s_k\}$ be a generating set for a finite abelian group A. Let $S' = \{s_1, s_2, \ldots, s_{k-1}\}$, $\langle S' \rangle = A' \leqslant A$, and $J = \langle s_k \rangle$. If $A' \cap J = \{0\}$, then*

$$\textsc{Cay}(A, S) \cong \textsc{Cay}(A', S') \,\square\, \textsc{Cay}(J, \langle s_k \rangle).$$

Lemma 6.1.3. *If $\Gamma = \textsc{Cay}(A, S)$ is a connected, 6-regular, abelian Cayley graph with a minimal connection set of involutions, then Γ has a Hamilton decomposition.*

Proof. As each element generates a 1-factor of Γ, we have $S = \{s_1, s_2, s_3, s_4, s_5, s_6\}$, where $|s_i| = 2$ for $1 \leqslant i \leqslant 6$. We will assume that $s_i \neq s_j$, for all $i \neq j$. Now, every element in A must have order 2, and so A is isomorphic to an elementary abelian 2-group. By repeatedly applying Lemma 6.1.2, we see

$$\Gamma \cong K_2 \,\square\, K_2 \,\square\, K_2 \,\square\, K_2 \,\square\, K_2 \,\square\, K_2 = C_4 \,\square\, C_4 \,\square\, C_4.$$

By Theorem 1.3.7, Γ is decomposable into three Hamilton cycles. ■

6.2 Fundamental Questions

We close by offering the following is a list of fundamental questions about Cayley graphs.

1. What are necessary and sufficient conditions for two Cayley graphs on the same group to be isomorphic?
2. For which Cayley graphs is G_R a normal subgroup of automorphisms?
3. Do almost all Cayley graphs have automorphism group as small as possible?
4. Are all nontrivial, connected, Cayley graphs hamiltonian? (Not even known for dihedral Cayley graphs, see [6].)
5. Which circulant digraphs are hamiltonian? ($\overrightarrow{\textsc{Cay}}(\mathbb{Z}_{12}, \{3, 4\})$ is not.)
6. Is every hamiltonian Cayley graph also edge-hamiltonian?
7. What Cayley graphs are Hamilton decomposable? (see Li-Yao [31].) In particular, are all Cayley graphs of p-groups Hamilton decomposable?

Appendix A

Data

This section contains Hamilton decompositions for certain 6-regular Cayley graphs (up to isomorphism) on abelian groups of orders 12, 18, 24, and 32, with a corresponding 3-element connection set that are not covered by previous theorems. The data was obtained via MAGMA programs and a random backtracking algorithm that finds Hamilton cycles (see Chapter B for source code).

1.1 Abelian Groups of Order 12

Table 1.1: Hamilton decompositions for Cayley graphs of order 12.

$\mathrm{Cay}(\mathbb{Z}_{12},\{2,3,4\})$
H_1:9,6,2,0,4,1,5,3,11,8,10,7
H_2:4,7,3,0,8,5,2,11,9,1,10,6
H_3:1,3,6,8,4,2,10,0,9,5,7,11
$\mathrm{Cay}(\mathbb{Z}_2\times\mathbb{Z}_6,\{(0,1),(0,2),(1,1)\})$
H_0:(1,1),(1,0),(0,1),(0,0),(0,4),(0,2),(0,3),(0,5),(1,4),(1,2),(1,3),(1,5)
H_1:(1,2),(0,3),(0,1),(0,2),(0,0),(1,5),(1,4),(1,0),(0,5),(0,4),(1,3),(1,1)
H_2:(0,1),(1,2),(1,0),(1,5),(0,4),(0,3),(1,4),(1,3),(0,2),(1,1),(0,0),(0,5)
$\mathrm{Cay}(\mathbb{Z}_2\times\mathbb{Z}_6,\{(0,1),(1,1),(1,2)\})$
H_0:(0,2),(1,4),(1,5),(0,0),(1,2),(0,3),(0,4),(1,0),(1,1),(0,5),(1,3),(0,1)
H_1:(1,0),(0,2),(0,3),(1,4),(1,3),(1,2),(1,1),(0,0),(0,5),(0,4),(1,5),(0,1)
H_2:(1,0),(0,5),(1,4),(0,0),(0,1),(1,2),(0,4),(1,3),(0,2),(1,1),(0,3),(1,5)

1.2 Abelian Groups of Order 18

Table 1.2: Hamilton decompositions for Cayley graphs of order 18.

$\mathrm{Cay}(\mathbb{Z}_{18},\{2,3,4\})$
H_1:0,14,12,9,7,3,6,4,1,17,15,11,13,10,8,5,2,16
H_2:5,3,17,14,10,12,16,1,15,13,9,6,2,0,4,8,11,7
H_3:4,2,17,13,16,14,11,9,5,1,3,0,15,12,8,6,10,7

Continued on next page

Table 1.2 – continued from previous page

$\text{Cay}(\mathbb{Z}_{18},\{3,4,6\})$
H_1:11,5,9,3,0,4,10,6,2,8,14,17,13,16,12,15,1,7
H_2:4,8,11,17,2,5,1,13,7,3,6,12,9,15,0,14,10,16
H_3:9,6,0,12,8,5,17,3,15,11,14,2,16,1,4,7,10,13
$\text{Cay}(\mathbb{Z}_3\times\mathbb{Z}_6,\{(1,3),(0,1),(1,2)\})$
H_0:(1,5),(1,0),(2,2),(2,1),(2,0),(1,3),(0,0),(0,1),(2,5),(2,4),(2,3),(1,1),(1,2),(0,5),(0,4),(0,3),(0,2),(1,4)
H_1:(2,1),(0,3),(1,0),(0,4),(1,1),(2,4),(0,0),(2,3),(0,5),(2,2),(1,5),(0,2),(2,0),(2,5),(1,2),(1,3),(0,1),(1,4)
H_2:(2,0),(0,3),(1,5),(2,1),(0,4),(2,2),(2,3),(1,0),(1,1),(0,5),(0,0),(1,2),(2,4),(0,1),(0,2),(2,5),(1,3),(1,4)
$\text{Cay}(\mathbb{Z}_3\times\mathbb{Z}_6,\{(1,3),(2,1),(1,1)\})$
H_0:(0,0),(2,1),(0,4),(1,1),(2,0),(0,1),(2,2),(0,5),(1,2),(0,3),(1,0),(2,3),(1,4),(2,5),(0,2),(1,5),(2,4),(1,3)
H_1:(0,3),(2,2),(1,5),(0,0),(2,3),(0,2),(1,1),(2,4),(0,1),(1,2),(2,1),(1,4),(0,5),(1,0),(2,5),(0,4),(1,3),(2,0)
H_2:(0,4),(1,5),(2,0),(0,5),(2,4),(0,3),(1,4),(0,1),(1,0),(2,1),(0,2),(1,3),(2,2),(1,1),(0,0),(2,5),(1,2),(2,3)
$\text{Cay}(\mathbb{Z}_3\times\mathbb{Z}_6,\{(0,1),(2,2),(1,0)\})$
H_0:(0,0),(1,0),(2,0),(0,4),(0,5),(2,1),(0,1),(1,5),(1,4),(1,3),(1,2),(0,2),(0,3),(1,1),(2,5),(2,4),(2,3),(2,2)
H_1:(1,3),(2,3),(0,1),(0,0),(2,0),(1,2),(1,1),(1,0),(1,5),(0,5),(2,5),(0,3),(0,4),(1,4),(2,4),(0,2),(2,2),(2,1)
H_2:(0,4),(1,2),(2,2),(1,4),(0,0),(0,5),(1,3),(0,3),(2,3),(1,5),(2,5),(2,0),(2,1),(1,1),(0,1),(0,2),(1,0),(2,4)
$\text{Cay}(\mathbb{Z}_3\times\mathbb{Z}_6,\{(2,1),(1,1),(1,2)\})$
H_0:(0,1),(2,0),(1,4),(0,5),(1,0),(2,2),(1,1),(0,2),(2,1),(0,0),(1,2),(2,3),(0,4),(1,5),(0,3),(2,4),(1,3),(2,5)
H_1:(0,1),(1,3),(0,2),(1,4),(0,3),(2,1),(1,2),(2,4),(0,5),(2,3),(1,1),(2,0),(1,5),(0,0),(2,5),(1,0),(0,4),(2,2)
H_2:(0,4),(2,5),(1,4),(2,3),(0,2),(2,0),(0,5),(1,1),(0,0),(2,4),(1,5),(2,1),(1,0),(0,1),(1,2),(0,3),(2,2),(1,3)
$\text{Cay}(\mathbb{Z}_3\times\mathbb{Z}_6,\{(2,1),(1,0),(1,2)\})$
H_0:(0,5),(1,4),(0,2),(2,0),(1,0),(0,0),(2,4),(0,3),(2,1),(1,2),(2,2),(0,4),(1,3),(2,3),(1,1),(0,1),(2,5),(1,5)
H_1:(0,0),(1,2),(2,4),(0,4),(1,0),(2,2),(0,1),(2,1),(1,1),(0,2),(2,3),(1,4),(2,0),(0,5),(2,5),(1,3),(0,3),(1,5)
H_2:(2,5),(1,0),(0,1),(1,3),(2,2),(0,2),(1,2),(0,3),(2,3),(0,5),(1,1),(2,0),(0,0),(2,1),(1,5),(2,4),(1,4),(0,4)

1.3 Abelian Groups of Order 24

Table 1.3: Hamilton decompositions for circulant graphs of order 24.

$\text{Cay}(\mathbb{Z}_{24},\{3,9,10\})$
H_0:6,9,18,4,7,16,2,5,8,22,19,10,1,11,14,17,20,23,13,3,0,15,12,21
H_1:15,1,16,19,5,14,4,13,10,0,21,7,22,12,9,23,2,11,20,6,3,17,8,18
H_2:2,12,3,18,21,11,8,23,14,0,9,19,4,1,22,13,16,6,15,5,20,10,7,17
$\text{Cay}(\mathbb{Z}_{24},\{2,6,9\})$
H_0:15,0,6,4,10,1,3,9,7,22,20,14,16,18,12,21,19,17,2,8,23,5,11,13
H_1:15,6,12,10,16,7,1,19,4,13,22,0,18,9,11,2,20,5,3,21,23,14,8,17
H_2:12,3,18,20,11,17,23,1,16,22,4,2,0,9,15,21,6,8,10,19,13,7,5,14
$\text{Cay}(\mathbb{Z}_{24},\{3,4,10\})$
H_0:19,5,9,13,17,20,6,10,14,0,4,18,15,11,1,21,7,3,23,2,16,12,8,22
H_1:8,5,1,15,12,2,22,18,14,17,3,6,9,19,16,13,23,20,10,0,21,11,7,4
H_2:10,13,3,0,20,16,6,2,5,15,19,23,9,12,22,1,4,14,11,8,18,21,17,7
$\text{Cay}(\mathbb{Z}_{24},\{2,9,10\})$
H_0:7,21,6,20,10,8,17,3,13,11,1,15,0,9,23,14,5,19,4,18,16,2,12,22
H_1:13,15,6,8,22,0,10,1,16,7,17,19,9,18,3,5,20,11,2,4,14,12,21,23
H_2:0,2,17,15,5,7,9,11,21,19,10,12,3,1,23,8,18,20,22,13,4,6,16,14
$\text{Cay}(\mathbb{Z}_{24},\{6,8,9\})$
H_0:0,9,1,10,4,20,5,23,14,8,16,7,13,22,6,15,21,12,3,19,11,17,2,18
H_1:3,18,10,2,8,0,6,12,4,19,13,21,5,11,20,14,22,16,1,7,15,23,17,9
H_2:3,11,2,20,12,18,9,15,0,16,10,19,1,17,8,23,7,22,4,13,5,14,6,21

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Table 1.3 – continued from previous page

$\mathrm{CAY}(\mathbb{Z}_{24},\{3,4,9\})$
H_0:5,20,0,3,12,15,6,2,23,19,10,7,4,8,11,14,17,21,1,16,13,22,18,9
H_1:3,6,21,18,14,23,8,5,1,10,13,17,20,16,12,9,0,4,19,15,11,2,22,7
H_2:22,1,4,13,9,6,10,14,5,2,17,8,12,21,0,15,18,3,23,20,11,7,16,19
$\mathrm{CAY}(\mathbb{Z}_{24},\{2,3,8\})$
H_0:19,22,6,3,1,4,2,18,16,14,17,9,11,13,15,23,20,12,10,7,5,8,0,21
H_1:16,0,2,5,3,11,14,6,8,10,13,21,23,7,4,12,9,1,22,20,18,15,17,19
H_2:18,10,2,23,1,17,20,4,6,9,7,15,12,14,22,0,3,19,11,8,16,13,5,21
$\mathrm{CAY}(\mathbb{Z}_{24},\{3,8,9\})$
H_0:2,18,15,0,3,11,20,12,21,6,9,17,14,5,13,10,1,22,7,4,19,16,8,23
H_1:5,2,17,8,0,9,1,4,12,3,6,22,14,11,19,10,18,21,13,16,7,15,23,20
H_2:0,16,1,17,20,4,13,22,19,3,18,9,12,15,6,14,23,7,10,2,11,8,5,21
$\mathrm{CAY}(\mathbb{Z}_{24},\{4,8,9\})$
H_0:6,21,13,4,8,12,3,7,22,2,10,18,9,17,1,16,0,15,23,19,11,20,5,14
H_1:10,19,15,7,11,2,17,13,5,21,1,9,0,4,12,20,16,8,23,3,18,14,22,6
H_2:7,16,12,21,17,8,0,20,4,19,3,11,15,6,2,18,22,13,9,5,1,10,14,23
$\mathrm{CAY}(\mathbb{Z}_{24},\{4,6,9\})$
H_0:0,18,3,23,8,12,16,1,19,10,4,22,2,6,15,21,17,11,7,13,9,5,14,20
H_1:16,7,22,18,12,3,9,15,11,5,23,14,10,1,21,6,0,4,19,13,17,8,2,20
H_2:23,17,2,11,20,5,1,7,3,21,12,6,10,16,22,13,4,8,14,18,9,0,15,19

Table 1.4: Hamilton decompositions for Cayley graphs on $\mathbb{Z}_2\times\mathbb{Z}_{12}$.

$\mathrm{CAY}(\mathbb{Z}_2\times\mathbb{Z}_{12},\{(1,4),(1,5),(0,3)\})$
H_0:(0,2),(1,6),(0,11),(1,4),(1,1),(0,8),(1,3),(1,0),(1,9),(0,1),(0,4),(1,8),(0,3),(1,7),(0,0),(0,9),(1,5),(0,10),(1,2),(0,7),(1,11),(0,6),(1,10),(0,5)
H_1:(0,3),(1,11),(0,4),(0,7),(1,3),(1,6),(0,10),(0,1),(1,5),(1,8),(0,0),(1,4),(0,8),(1,0),(0,5),(1,9),(0,2),(0,11),(1,7),(1,10),(1,1),(0,9),(1,2),(0,6)
H_2:(0,1),(1,8),(1,11),(1,2),(1,5),(0,0),(0,3),(1,10),(0,2),(1,7),(1,4),(0,9),(0,6),(1,1),(0,5),(0,8),(0,11),(1,3),(0,10),(0,7),(1,0),(0,4),(1,9),(1,6)
$\mathrm{CAY}(\mathbb{Z}_2\times\mathbb{Z}_{12},\{(1,3),(1,4),(0,3)\})$
H_0:(0,8),(0,5),(1,2),(0,6),(0,3),(1,7),(0,11),(1,3),(1,0),(1,9),(0,0),(1,4),(0,1),(0,4),(1,1),(0,10),(0,7),(1,10),(0,2),(1,6),(0,9),(1,5),(1,8),(1,11)
H_1:(1,8),(0,11),(0,2),(0,5),(1,1),(0,9),(0,0),(0,3),(1,11),(1,2),(1,5),(0,1),(0,10),(1,7),(1,10),(0,6),(1,9),(1,6),(1,3),(0,7),(1,4),(0,8),(1,0),(0,4)
H_2:(0,1),(1,9),(0,5),(1,8),(0,0),(1,3),(0,6),(0,9),(1,0),(0,3),(1,6),(0,10),(1,2),(0,11),(0,8),(1,5),(0,2),(1,11),(0,7),(0,4),(1,7),(1,4),(1,1),(1,10)
$\mathrm{CAY}(\mathbb{Z}_2\times\mathbb{Z}_{12},\{(0,1),(0,2),(1,1)\})$
H_0:(0,6),(0,7),(1,8),(1,10),(1,0),(1,1),(0,2),(0,4),(0,3),(0,1),(1,2),(1,3),(1,4),(0,5),(1,6),(1,5),(1,7),(1,9),(0,10),(1,11),(0,0),(0,11),(0,9),(0,8)
H_1:(1,3),(0,2),(0,0),(0,1),(1,0),(0,11),(1,10),(0,9),(0,10),(0,8),(0,7),(1,6),(1,7),(1,8),(1,9),(1,11),(1,1),(1,2),(1,4),(0,3),(0,5),(0,4),(0,6),(1,5)
H_2:(1,9),(0,8),(1,7),(0,6),(0,5),(0,7),(0,9),(1,8),(1,6),(1,4),(1,5),(0,4),(1,3),(1,1),(0,0),(0,10),(0,11),(0,1),(0,2),(0,3),(1,2),(1,0),(1,11),(1,10)
$\mathrm{CAY}(\mathbb{Z}_2\times\mathbb{Z}_{12},\{(1,3),(0,4),(1,2)\})$
H_0:(0,1),(1,4),(0,6),(1,8),(1,0),(0,2),(1,11),(0,9),(1,6),(0,4),(0,0),(1,3),(1,7),(0,10),(1,1),(1,9),(0,7),(0,11),(0,3),(1,5),(0,8),(1,10),(1,2),(0,5)
H_1:(0,1),(1,3),(0,5),(1,8),(1,4),(0,7),(0,3),(1,1),(0,4),(1,7),(0,9),(1,0),(0,10),(0,6),(0,2),(1,5),(1,9),(0,11),(1,2),(1,6),(1,10),(0,0),(0,8),(1,11)
H_2:(0,5),(1,7),(1,11),(1,3),(0,6),(1,9),(0,0),(1,2),(0,4),(0,8),(1,6),(0,3),(1,0),(1,4),(0,2),(0,10),(1,8),(0,11),(1,1),(1,5),

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Table 1.4 – continued from previous page

(0,7),(1,10),(0,1),(0,9)

$\textsc{Cay}(\mathbb{Z}_2\times\mathbb{Z}_{12},\{(1,3),(1,5),(0,3)\})$

H_0:(0,4),(0,1),(1,4),(1,7),(0,2),(1,5),(0,8),(1,11),(1,2),(0,5),(1,8),(0,11),(1,6),(1,3),(0,6),(0,9),(1,0),(0,7),(0,10),(1,1),(1,10),(0,3),(0,0),(1,9)

H_1:(0,7),(1,2),(0,9),(1,6),(0,3),(0,6),(1,1),(0,4),(1,7),(1,10),(0,1),(0,10),(1,3),(0,0),(1,5),(1,8),(1,11),(0,2),(1,9),(1,0),(0,5),(0,8),(0,11),(1,4)

H_2:(0,8),(1,3),(1,0),(0,3),(1,8),(0,1),(1,6),(1,9),(0,6),(1,11),(0,4),(0,7),(1,10),(0,5),(0,2),(0,11),(1,2),(1,5),(0,10),(1,7),(0,0),(0,9),(1,4),(1,1)

$\textsc{Cay}(\mathbb{Z}_2\times\mathbb{Z}_{12},\{(0,4),(0,5),(1,2)\})$

H_0:(0,3),(0,8),(0,0),(0,4),(0,11),(0,6),(1,8),(0,10),(1,0),(0,2),(0,9),(1,7),(1,2),(1,9),(1,4),(1,11),(0,1),(0,5),(1,3),(1,10),(1,6),(1,1),(1,5),(0,7)

H_1:(0,1),(1,3),(1,7),(1,0),(1,4),(1,8),(1,1),(0,3),(1,5),(1,9),(0,11),(0,7),(0,2),(0,6),(0,10),(0,5),(0,0),(1,2),(1,10),(0,8),(0,4),(1,6),(1,11),(0,9)

H_2:(1,8),(1,0),(1,5),(1,10),(0,0),(0,7),(1,9),(1,1),(0,11),(0,3),(0,10),(0,2),(1,4),(0,6),(0,1),(0,8),(1,6),(1,2),(0,4),(0,9),(0,5),(1,7),(1,11),(1,3)

$\textsc{Cay}(\mathbb{Z}_2\times\mathbb{Z}_{12},\{(0,2),(1,5),(1,2)\})$

H_0:(1,2),(0,7),(1,0),(0,5),(0,3),(1,8),(1,6),(0,4),(0,2),(0,0),(1,10),(0,8),(0,6),(1,1),(1,3),(0,10),(1,5),(1,7),(1,9),(0,11),(0,1),(1,11),(0,9),(1,4)

H_1:(1,11),(0,6),(1,4),(1,6),(0,8),(1,3),(0,5),(1,7),(0,2),(1,0),(1,10),(0,3),(0,1),(1,8),(0,10),(0,0),(1,5),(0,7),(1,9),(0,4),(1,2),(0,9),(0,11),(1,1)

H_2:(1,1),(0,3),(1,5),(1,3),(0,1),(1,6),(0,11),(1,4),(0,2),(1,9),(1,11),(0,4),(0,6),(1,8),(1,10),(0,5),(0,7),(0,9),(1,7),(0,0),(1,2),(1,0),(0,10),(0,8)

$\textsc{Cay}(\mathbb{Z}_2\times\mathbb{Z}_{12},\{(1,3),(0,1),(1,5)\})$

H_0:(0,3),(1,6),(0,1),(0,2),(1,11),(1,10),(1,9),(0,6),(0,5),(0,4),(1,7),(0,0),(1,3),(1,4),(0,7),(1,0),(1,1),(0,8),(1,5),(0,10),(0,9),(1,2),(0,11),(1,8)

H_1:(1,0),(0,3),(0,2),(1,9),(0,4),(1,1),(1,2),(0,7),(0,6),(1,3),(0,10),(0,11),(0,0),(0,1),(1,10),(0,5),(1,8),(1,7),(1,6),(1,5),(1,4),(0,9),(0,8),(1,11)

H_2:(0,5),(1,2),(1,3),(0,8),(0,7),(1,10),(0,3),(0,4),(1,11),(0,6),(1,1),(0,10),(1,7),(0,2),(1,5),(0,0),(1,9),(1,8),(0,1),(1,4),(0,11),(1,6),(0,9),(1,0)

$\textsc{Cay}(\mathbb{Z}_2\times\mathbb{Z}_{12},\{(0,1),(0,3),(1,2)\})$

H_0:(1,9),(0,7),(1,5),(1,2),(1,11),(1,0),(0,10),(1,8),(1,7),(1,4),(0,2),(0,11),(1,1),(0,3),(0,0),(1,10),(0,8),(0,9),(0,6),(0,5),(1,3),(0,1),(0,4),(1,6)

H_1:(1,5),(1,8),(0,6),(0,7),(0,4),(0,3),(0,2),(0,5),(1,7),(1,6),(0,8),(0,11),(0,0),(0,1),(0,10),(0,9),(1,11),(1,10),(1,9),(1,0),(1,1),(1,2),(1,3),(1,4)

H_2:(1,6),(1,3),(1,0),(0,2),(0,1),(1,11),(1,8),(1,9),(0,11),(0,10),(0,7),(0,8),(0,5),(0,4),(1,2),(0,0),(0,9),(1,7),(1,10),(1,1),(1,4),(0,6),(0,3),(1,5)

$\textsc{Cay}(\mathbb{Z}_2\times\mathbb{Z}_{12},\{(1,3),(0,1),(0,4)\})$

H_0:(1,1),(1,2),(0,5),(0,1),(1,4),(1,5),(0,8),(0,7),(0,6),(1,9),(0,0),(1,3),(1,11),(1,10),(1,6),(0,9),(1,0),(0,3),(0,2),(0,10),(0,11),(1,8),(1,7),(0,4)

H_1:(0,10),(1,1),(1,5),(0,2),(0,1),(0,9),(0,5),(0,4),(0,3),(1,6),(1,2),(1,10),(1,9),(1,8),(1,0),(1,4),(0,7),(0,11),(0,0),(0,8),(1,11),(1,7),(1,3),(0,6)

H_2:(1,5),(1,6),(1,7),(0,10),(0,9),(0,8),(0,4),(0,0),(0,1),(1,10),(0,7),(0,3),(0,11),(1,2),(1,3),(1,4),(1,8),(0,5),(0,6),(0,2),(1,11),(1,0),(1,1),(1,9)

$\textsc{Cay}(\mathbb{Z}_2\times\mathbb{Z}_{12},\{(0,2),(0,3),(1,2)\})$

H_0:(0,0),(0,3),(0,1),(1,3),(1,5),(1,2),(0,4),(0,2),(1,4),(1,1),(0,11),(0,9),(0,6),(0,8),(1,6),(1,8),(1,11),(1,9),(1,7),(0,5),(0,7),(0,10),(1,0),(1,10)

H_1:(1,5),(0,3),(0,6),(0,4),(0,7),(0,9),(0,0),(0,2),(1,0),(1,9),(1,6),(1,4),(1,2),(1,11),(1,1),(1,3),(0,5),(0,8),(0,11),(0,1),(0,10),(1,8),(1,10),(1,7)

H_2:(1,8),(1,5),(0,7),(1,9),(0,11),(0,2),(0,5),(0,3),(1,1),(1,10),(0,8),(0,10),(0,0),(1,2),(1,0),(1,3),(1,6),(0,4),(0,1),(1,11),(0,9),(1,7),(1,4),(0,6)

$\textsc{Cay}(\mathbb{Z}_2\times\mathbb{Z}_{12},\{(1,4),(1,1),(1,2)\})$

Continued on next page

Table 1.4 – continued from previous page

H_0:(1,6),(0,4),(1,2),(0,6),(1,7),(0,5),(1,3),(0,2),(1,4),(0,3),(1,1),(0,9),(1,5),(0,7),(1,8),(0,10),(1,9),(0,1),(1,11),(0,0),(1,10),(0,11),(1,0),(0,8)

H_1:(0,3),(1,11),(0,7),(1,3),(0,4),(1,8),(0,6),(1,5),(0,1),(1,0),(0,10),(1,6),(0,5),(1,9),(0,11),(1,1),(0,2),(1,10),(0,9),(1,7),(0,8),(1,4),(0,0),(1,2)

H_2:(0,0),(1,1),(0,5),(1,4),(0,6),(1,10),(0,8),(1,9),(0,7),(1,6),(0,2),(1,0),(0,4),(1,5),(0,3),(1,7),(0,11),(1,3),(0,1),(1,2),(0,10),(1,11),(0,9),(1,8)

$\textsc{Cay}(\mathbb{Z}_2{\times}\mathbb{Z}_{12},\{(1,4),(1,5),(0,5)\})$

H_0:(1,9),(1,2),(0,7),(0,2),(1,10),(0,3),(1,8),(0,0),(0,5),(1,0),(1,7),(0,11),(0,4),(1,11),(0,6),(1,1),(0,8),(1,3),(0,10),(1,6),(0,1),(1,5),(0,9),(1,4)

H_1:(1,4),(0,8),(1,0),(0,7),(0,0),(1,5),(0,10),(1,2),(0,6),(1,10),(1,3),(0,11),(1,6),(1,1),(0,5),(1,9),(0,1),(1,8),(0,4),(0,9),(0,2),(1,7),(0,3),(1,11)

H_2:(1,9),(0,2),(1,6),(1,11),(0,7),(1,3),(1,8),(1,1),(0,9),(1,2),(1,7),(0,0),(1,4),(0,11),(0,6),(0,1),(0,8),(0,3),(0,10),(0,5),(1,10),(1,5),(1,0),(0,4)

$\textsc{Cay}(\mathbb{Z}_2{\times}\mathbb{Z}_{12},\{(1,3),(0,1),(0,5)\})$

H_0:(1,4),(0,1),(0,2),(0,9),(1,6),(1,5),(1,0),(1,1),(1,2),(1,3),(0,0),(0,5),(0,6),(0,11),(0,4),(0,3),(0,10),(1,7),(1,8),(1,9),(1,10),(0,7),(0,8),(1,11)

H_1:(1,0),(1,7),(1,2),(0,11),(0,0),(0,1),(0,6),(1,9),(1,4),(0,7),(0,2),(0,3),(0,8),(1,5),(1,10),(1,3),(1,8),(0,5),(0,4),(0,9),(0,10),(1,1),(1,6),(1,11)

H_2:(1,10),(0,1),(0,8),(0,9),(1,0),(0,3),(1,6),(1,7),(0,4),(1,1),(1,8),(0,11),(0,10),(0,5),(1,2),(1,9),(0,0),(0,7),(0,6),(1,3),(1,4),(1,5),(0,2),(1,11)

$\textsc{Cay}(\mathbb{Z}_2{\times}\mathbb{Z}_{12},\{(0,2),(0,3),(1,1)\})$

H_0:(0,7),(0,4),(1,3),(0,2),(0,5),(0,3),(1,2),(1,5),(0,6),(1,7),(1,10),(1,0),(0,11),(0,1),(0,10),(0,8),(1,9),(1,11),(0,0),(1,1),(1,4),(1,6),(1,8),(0,9)

H_1:(1,2),(1,4),(0,3),(0,6),(0,8),(1,7),(1,9),(1,6),(0,5),(0,7),(1,8),(1,5),(1,3),(1,0),(0,1),(0,4),(0,2),(1,1),(1,10),(0,11),(0,9),(0,0),(0,10),(1,11)

H_2:(0,9),(0,6),(0,4),(1,5),(1,7),(1,4),(0,5),(0,8),(0,11),(0,2),(0,0),(0,3),(0,1),(1,2),(1,0),(1,9),(0,10),(0,7),(1,6),(1,3),(1,1),(1,11),(1,8),(1,10)

$\textsc{Cay}(\mathbb{Z}_2{\times}\mathbb{Z}_{12},\{(0,1),(0,4),(1,1)\})$

H_0:(1,2),(0,3),(1,4),(1,3),(0,4),(0,5),(0,1),(1,0),(1,1),(0,0),(0,8),(1,7),(1,11),(1,10),(0,9),(1,8),(1,9),(1,5),(0,6),(0,2),(0,10),(0,11),(0,7),(1,6)

H_1:(1,1),(0,2),(0,3),(0,7),(0,6),(1,7),(1,6),(0,5),(0,9),(0,1),(0,0),(0,4),(0,8),(1,9),(0,10),(1,11),(1,3),(1,2),(1,10),(0,11),(1,0),(1,8),(1,4),(1,5)

H_2:(1,11),(0,0),(0,11),(0,3),(0,4),(1,5),(1,6),(1,10),(1,9),(1,1),(1,2),(0,1),(0,2),(1,3),(1,7),(1,8),(0,7),(0,8),(0,9),(0,10),(0,6),(0,5),(1,4),(1,0)

$\textsc{Cay}(\mathbb{Z}_2{\times}\mathbb{Z}_{12},\{(0,1),(1,4),(0,5)\})$

H_0:(0,6),(0,1),(0,8),(0,3),(1,11),(0,7),(0,2),(1,6),(1,5),(1,10),(1,3),(0,11),(0,0),(1,4),(1,9),(0,5),(1,1),(1,8),(1,7),(1,0),(0,4),(0,9),(0,10),(1,2)

H_1:(0,4),(0,3),(0,2),(0,9),(1,5),(0,1),(0,0),(0,5),(0,6),(0,7),(1,3),(1,2),(1,7),(0,11),(0,10),(1,6),(1,1),(1,0),(0,8),(1,4),(1,11),(1,10),(1,9),(1,8)

H_2:(0,2),(0,1),(1,9),(1,2),(1,1),(0,9),(0,8),(0,7),(0,0),(1,8),(1,3),(1,4),(1,5),(1,0),(1,11),(1,6),(1,7),(0,3),(0,10),(0,5),(0,4),(0,11),(0,6),(1,10)

$\textsc{Cay}(\mathbb{Z}_2{\times}\mathbb{Z}_{12},\{(1,3),(0,3),(0,4)\})$

H_0:(1,6),(1,2),(0,5),(0,1),(0,10),(1,1),(1,4),(1,8),(1,0),(0,3),(0,7),(0,11),(0,2),(1,11),(1,3),(0,6),(0,9),(0,0),(1,9),(1,5),(0,8),(0,4),(1,7),(1,10)

H_1:(0,5),(0,8),(0,11),(1,2),(1,5),(1,8),(1,11),(1,7),(1,3),(1,0),(1,4),(0,1),(0,4),(0,0),(0,3),(0,6),(0,2),(0,10),(0,7),(1,10),(1,1),(1,9),(1,6),(0,9)

H_2:(1,3),(1,6),(0,3),(0,11),(1,8),(0,5),(0,2),(1,5),(1,1),(0,4),(0,7),(1,4),(1,7),(0,10),(0,6),(1,9),(1,0),(0,9),(0,1),(1,10),(1,2),(1,11),(0,8),(0,0)

Table 1.5: Hamilton decompositions for Cayley graphs on $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_6$.

$\text{Cay}(\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_6,\{(0,0,1),(1,1,1),(1,0,1)\})$
H_0:(0,0,4),(0,0,5),(0,0,0),(0,0,1),(1,1,2),(0,1,3),(0,1,4),(1,0,3),(0,0,2),(1,0,1),(0,1,0),(0,1,1),(1,0,2),(0,0,3),(1,0,4), (1,0,5),(1,0,0),(0,1,5),(1,1,0),(1,1,1),(0,1,2),(1,1,3),(1,1,4),(1,1,5)
H_1:(1,0,3),(0,0,4),(1,1,3),(0,1,4),(1,1,5),(0,0,0),(1,0,5),(0,1,0),(0,1,5),(1,0,4),(0,0,5),(1,0,0),(0,0,1),(1,1,0),(0,1,1), (1,1,2),(1,1,1),(0,0,2),(0,0,3),(1,1,4),(0,1,3),(1,0,2),(1,0,1),(0,1,2)
H_2:(0,1,1),(0,1,2),(0,1,3),(1,0,4),(1,0,3),(1,0,2),(0,0,1),(0,0,2),(1,1,3),(1,1,2),(0,0,3),(0,0,4),(1,0,5),(0,1,4),(0,1,5), (1,1,4),(0,0,5),(1,1,0),(1,1,5),(0,1,0),(1,1,1),(0,0,0),(1,0,1),(1,0,0)

1.4 Abelian Groups of Order 32

Table 1.6: Hamilton decompositions for Cayley graphs on $\mathbb{Z}_2\times\mathbb{Z}_{16}$.

$\text{Cay}(\mathbb{Z}_2\times\mathbb{Z}_{16},\{(0,2),(1,5),(0,3)\})$
H_0:(1,7),(1,4),(1,6),(1,3),(1,5),(0,10),(0,8),(0,6),(0,3),(0,1),(0,4),(0,7),(0,9),(0,11),(0,13),(0,0),(0,14),(0,12),(0,15), (0,2),(0,5),(1,10),(1,8),(1,11),(1,13),(1,0),(1,2),(1,15),(1,1),(1,14),(1,12),(1,9)
H_1:(1,6),(0,1),(1,12),(1,10),(1,13),(0,8),(0,5),(1,0),(0,11),(0,14),(1,3),(1,1),(1,4),(0,15),(0,13),(1,2),(0,7),(0,10),(1,15), (0,4),(0,6),(0,9),(0,12),(1,7),(0,2),(0,0),(1,5),(1,8),(0,3),(1,14),(1,11),(1,9)
H_2:(0,3),(0,0),(1,11),(0,6),(1,1),(0,12),(0,10),(0,13),(1,8),(1,6),(0,11),(0,8),(1,3),(1,0),(1,14),(0,9),(1,4),(1,2),(1,5), (1,7),(1,10),(0,15),(0,1),(0,14),(1,9),(0,4),(0,2),(1,13),(1,15),(1,12),(0,7),(0,5)
$\text{Cay}(\mathbb{Z}_2\times\mathbb{Z}_{16},\{(1,5),(1,6),(1,1)\})$
H_0:(0,2),(1,8),(0,9),(1,14),(0,3),(1,9),(0,8),(1,2),(0,13),(1,7),(0,1),(1,11),(0,6),(1,1),(0,0),(1,10),(0,4),(1,5),(0,10), (1,4),(0,15),(1,0),(0,11),(1,12),(0,7),(1,13),(0,12),(1,6),(0,5),(1,15),(0,14),(1,3)
H_1:(0,6),(1,5),(0,11),(1,10),(0,5),(1,0),(0,1),(1,6),(0,7),(1,1),(0,12),(1,2),(0,3),(1,8),(0,13),(1,3),(0,4),(1,14),(0,15), (1,9),(0,10),(1,11),(0,0),(1,15),(0,9),(1,4),(0,14),(1,13),(0,8),(1,7),(0,2),(1,12)
H_2:(1,15),(0,4),(1,9),(0,14),(1,8),(0,7),(1,2),(0,1),(1,12),(0,13),(1,14),(0,8),(1,3),(0,9),(1,10),(0,15),(1,5),(0,0),(1,6), (0,11),(1,1),(0,2),(1,13),(0,3),(1,4),(0,5),(1,11),(0,12),(1,7),(0,6),(1,0),(0,10)
$\text{Cay}(\mathbb{Z}_2\times\mathbb{Z}_{16},\{(1,3),(0,5),(1,2)\})$
H_0:(1,14),(0,1),(1,3),(0,5),(1,2),(0,0),(0,11),(1,8),(0,6),(1,9),(0,7),(0,2),(1,4),(1,15),(0,13),(0,8),(1,11),(0,9),(1,7), (0,4),(1,6),(0,3),(1,1),(0,15),(1,13),(0,10),(1,12),(0,14),(1,0),(1,5),(1,10),(0,12)
H_1:(0,1),(1,4),(0,6),(1,3),(0,0),(0,5),(1,7),(0,10),(1,8),(1,13),(0,11),(1,14),(1,9),(0,12),(0,7),(1,5),(0,8),(0,3),(0,14), (1,1),(1,12),(0,15),(1,2),(0,4),(0,9),(1,6),(1,11),(1,0),(0,2),(0,13),(1,10),(1,15)
H_2:(0,8),(1,6),(1,1),(0,4),(0,15),(0,10),(0,5),(1,8),(1,3),(1,14),(0,0),(1,13),(1,2),(1,7),(1,12),(0,9),(0,14),(1,11),(0,13), (1,0),(0,3),(1,5),(0,2),(1,15),(0,12),(0,1),(0,6),(0,11),(1,9),(1,4),(0,7),(1,10)
$\text{Cay}(\mathbb{Z}_2\times\mathbb{Z}_{16},\{(1,3),(0,7),(1,1)\})$
H_0:(1,6),(0,5),(1,2),(0,15),(1,12),(1,3),(0,4),(1,7),(1,0),(0,1),(1,4),(0,7),(0,14),(1,1),(0,2),(1,5),(0,8),(1,9),(0,6), (0,13),(1,14),(0,11),(1,10),(0,9),(1,8),(1,15),(0,0),(1,13),(0,10),(1,11),(0,12),(0,3)
H_1:(1,13),(0,12),(1,9),(1,0),(0,3),(1,2),(0,1),(0,10),(1,7),(0,8),(0,15),(1,14),(1,5),(0,6),(1,3),(0,2),(0,9),(0,0),(1,1), (0,4),(0,11),(1,12),(0,13),(1,10),(0,7),(1,8),(0,5),(1,4),(1,11),(0,14),(1,15),(1,6)
H_2:(0,6),(1,7),(1,14),(0,1),(0,8),(1,11),(1,2),(1,9),(0,10),(0,3),(1,4),(1,13),(0,14),(0,5),(0,12),(1,15),(0,2),(0,11),(1,8), (1,1),(1,10),(1,3),(0,0),(0,7),(1,6),(0,9),(1,12),(1,5),(0,4),(0,13),(1,0),(0,15)
$\text{Cay}(\mathbb{Z}_2\times\mathbb{Z}_{16},\{(1,3),(0,1),(1,5)\})$
H_0:(0,0),(0,15),(0,14),(1,1),(0,12),(1,15),(1,0),(0,11),(0,10),(1,7),(0,4),(0,3),(1,6),(0,9),(1,14),(1,13),(0,8),(0,7),(0,6), (0,5),(1,8),(1,9),(1,10),(0,13),(1,2),(1,3),(1,4),(1,5),(0,2),(0,1),(1,12),(1,11)
H_1:(0,5),(1,2),(0,7),(1,4),(0,1),(1,6),(1,7),(0,2),(0,3),(1,14),(0,11),(1,8),(0,13),(0,12),(1,9),(0,6),(1,3),(0,14),(1,11), (1,10),(0,15),(1,12),(0,9),(0,8),(1,5),(0,0),(1,13),(0,10),(1,15),(0,4),(1,1),(1,0)
H_2:(0,10),(1,5),(1,6),(0,11),(0,12),(1,7),(1,8),(0,3),(1,0),(0,13),(0,14),(1,9),(0,4),(0,5),(1,10),(0,7),(1,12),(1,13),(0,2), (1,15),(1,14),(0,1),(0,0),(1,3),(0,8),(1,11),(0,6),(1,1),(1,2),(0,15),(1,4),(0,9)
$\text{Cay}(\mathbb{Z}_2\times\mathbb{Z}_{16},\{(1,4),(0,3),(1,7)\})$
H_0:(1,10),(0,1),(1,5),(1,8),(0,4),(1,11),(0,7),(1,3),(0,12),(0,9),(1,0),(1,13),(0,6),(1,2),(0,11),(0,14),(1,7),(0,0),(1,4),

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Table 1.6 – continued from previous page

(0,13),(1,9),(0,5),(0,2),(0,15),(1,6),(0,10),(1,14),(1,1),(0,8),(1,12),(1,15),(0,3)

H_1:(0,2),(1,6),(1,9),(1,12),(0,0),(0,3),(0,6),(1,10),(1,7),(1,4),(1,1),(0,13),(0,10),(1,3),(0,15),(1,8),(0,12),(1,0),(0,7),(0,4),(0,1),(1,13),(0,9),(1,5),(0,14),(1,2),(1,15),(0,11),(0,8),(0,5),(1,14),(1,11)

H_2:(1,3),(1,0),(0,4),(1,13),(1,10),(0,14),(0,1),(1,8),(1,11),(0,15),(0,12),(1,5),(1,2),(0,9),(0,6),(1,15),(0,8),(1,4),(0,11),(1,7),(0,3),(1,12),(0,5),(1,1),(0,10),(0,7),(1,14),(0,2),(1,9),(0,0),(0,13),(1,6)

Cay($\mathbb{Z}_2\times\mathbb{Z}_{16}$,{(1,4),(0,2),(0,5)})

H_0:(0,5),(0,0),(0,2),(0,4),(0,6),(0,1),(0,3),(1,15),(1,1),(0,13),(1,9),(1,4),(1,6),(0,10),(0,8),(1,12),(1,7),(0,11),(0,9),(1,5),(1,0),(1,14),(1,3),(0,15),(1,11),(1,13),(1,2),(0,14),(1,10),(1,8),(0,12),(0,7)

H_1:(1,0),(1,2),(1,7),(0,3),(0,14),(0,12),(0,1),(0,15),(0,10),(0,5),(1,1),(1,3),(1,5),(1,10),(1,15),(1,13),(0,9),(0,7),(0,2),(0,13),(0,11),(0,6),(0,8),(1,4),(0,0),(1,12),(1,14),(1,9),(1,11),(1,6),(1,8),(0,4)

H_2:(1,14),(0,2),(1,6),(1,1),(1,12),(1,10),(0,6),(1,2),(1,4),(1,15),(0,11),(0,0),(0,14),(0,9),(0,4),(0,15),(0,13),(0,8),(0,3),(0,5),(1,9),(1,7),(1,5),(0,1),(1,13),(1,8),(1,3),(0,7),(1,11),(1,0),(0,12),(0,10)

Cay($\mathbb{Z}_2\times\mathbb{Z}_{16}$,{(0,1),(1,6),(0,7)})

H_0:(1,6),(0,0),(0,1),(0,2),(0,9),(1,3),(1,10),(1,1),(0,11),(0,4),(0,3),(1,9),(1,2),(0,8),(0,7),(0,6),(0,15),(1,5),(1,4),(0,10),(1,0),(1,7),(0,13),(0,14),(1,8),(1,15),(1,14),(1,13),(1,12),(1,11),(0,5),(0,12)

H_1:(1,4),(1,11),(0,1),(0,8),(1,14),(1,7),(1,6),(1,15),(1,0),(0,6),(0,5),(0,4),(0,13),(1,3),(1,2),(1,1),(1,8),(0,2),(1,12),(1,5),(0,11),(0,12),(0,3),(0,10),(0,9),(0,0),(1,10),(1,9),(0,15),(0,14),(0,7),(1,13)

H_2:(0,13),(0,6),(1,12),(1,3),(1,4),(0,14),(0,5),(1,15),(0,9),(0,8),(0,15),(0,0),(0,7),(1,1),(1,0),(1,9),(1,8),(1,7),(0,1),(0,10),(0,11),(0,2),(0,3),(1,13),(1,6),(1,5),(1,14),(0,4),(1,10),(1,11),(1,2),(0,12)

Cay($\mathbb{Z}_2\times\mathbb{Z}_{16}$,{(1,5),(0,4),(1,2)})

H_0:(0,14),(1,3),(0,1),(0,5),(1,10),(0,12),(0,0),(1,14),(0,3),(0,15),(1,1),(0,6),(1,8),(0,13),(1,2),(0,4),(1,15),(1,11),(1,7),(0,9),(1,4),(1,0),(0,2),(1,13),(0,8),(1,6),(0,11),(1,9),(1,5),(0,7),(1,12),(0,10)

H_1:(0,11),(0,15),(1,4),(0,6),(0,10),(1,8),(1,12),(0,14),(0,2),(1,7),(1,3),(0,8),(1,10),(1,14),(0,12),(1,1),(1,13),(1,9),(0,4),(0,0),(1,5),(0,3),(0,7),(1,2),(1,6),(0,1),(1,15),(0,13),(1,11),(0,9),(0,5),(1,0)

H_2:(1,9),(0,7),(0,11),(1,13),(0,15),(1,10),(1,6),(0,4),(0,8),(0,12),(1,7),(0,5),(1,3),(1,15),(0,10),(1,5),(1,1),(0,3),(1,8),(1,4),(0,2),(0,6),(1,11),(0,0),(1,2),(1,14),(0,9),(0,13),(0,1),(1,12),(1,0),(0,14)

Cay($\mathbb{Z}_2\times\mathbb{Z}_{16}$,{(0,1),(1,4),(1,2)})

H_0:(0,5),(1,1),(1,2),(0,4),(1,0),(1,15),(0,1),(0,2),(1,14),(0,0),(1,12),(1,13),(0,9),(1,5),(0,3),(1,7),(0,11),(1,9),(1,8),(0,10),(1,6),(0,8),(1,4),(1,3),(0,15),(0,14),(0,13),(0,12),(1,10),(1,11),(0,7),(0,6)

H_1:(1,1),(0,15),(0,0),(0,1),(1,5),(1,6),(0,2),(1,4),(0,6),(1,10),(1,9),(0,5),(0,4),(0,3),(1,15),(0,13),(1,11),(0,9),(1,7),(1,8),(0,12),(0,11),(1,13),(1,14),(0,10),(1,12),(0,8),(0,7),(1,3),(1,2),(0,14),(1,0)

H_2:(1,0),(0,2),(0,3),(1,1),(0,13),(1,9),(0,7),(1,5),(1,4),(0,0),(1,2),(0,6),(1,8),(0,4),(1,6),(1,7),(0,5),(1,3),(0,1),(1,13),(0,15),(1,11),(1,12),(0,14),(1,10),(0,8),(0,9),(0,10),(0,11),(1,15),(1,14),(0,12)

Cay($\mathbb{Z}_2\times\mathbb{Z}_{16}$,{(1,5),(1,6),(1,2)})

H_0:(1,3),(0,1),(1,15),(0,10),(1,8),(0,2),(1,12),(0,14),(1,0),(0,11),(1,13),(0,3),(1,1),(0,6),(1,4),(0,9),(1,14),(0,4),(1,2),(0,7),(1,9),(0,15),(1,5),(0,0),(1,6),(0,12),(1,7),(0,13),(1,11),(0,5),(1,10),(0,8)

H_1:(0,3),(1,5),(0,10),(1,4),(0,14),(1,3),(0,5),(1,7),(0,1),(1,11),(0,9),(1,15),(0,4),(1,9),(0,11),(1,6),(0,8),(1,14),(0,0),(1,10),(0,15),(1,13),(0,2),(1,0),(0,6),(1,12),(0,7),(1,1),(0,12),(1,2),(0,13),(1,8)

H_2:(0,6),(1,11),(0,0),(1,2),(0,8),(1,13),(0,7),(1,5),(0,11),(1,1),(0,15),(1,4),(0,2),(1,7),(0,9),(1,3),(0,13),(1,15),(0,5),(1,0),(0,10),(1,12),(0,1),(1,6),(0,4),(1,10),(0,12),(1,14),(0,3),(1,9),(0,14),(1,8)

Cay($\mathbb{Z}_2\times\mathbb{Z}_{16}$,{(1,3),(0,7),(1,2)})

H_0:(0,9),(0,2),(1,4),(0,1),(1,3),(0,0),(1,13),(0,11),(0,4),(0,13),(1,0),(1,7),(1,14),(0,12),(1,10),(0,8),(1,6),(1,15),(1,8),(1,1),(0,3),(0,10),(1,12),(1,5),(0,7),(1,9),(0,6),(0,15),(1,2),(0,5),(0,14),(1,11)

H_1:(0,1),(0,10),(1,7),(0,5),(1,3),(0,6),(1,8),(0,11),(1,14),(0,0),(1,2),(0,4),(1,6),(0,9),(1,12),(0,14),(0,7),(1,4),(1,13),(0,15),(1,1),(1,10),(0,13),(1,11),(0,8),(1,5),(0,3),(0,12),(1,9),(1,0),(0,2),(1,15)

H_2:(1,5),(0,2),(0,11),(1,9),(1,2),(1,11),(1,4),(0,6),(0,13),(1,15),(0,12),(0,5),(1,8),(0,10),(1,13),(1,6),(0,3),(1,0),(0,14),(1,1),(0,4),(1,7),(0,9),(0,0),(0,7),(1,10),(1,3),(1,12),(0,15),(0,8),(0,1),(1,14)

Cay($\mathbb{Z}_2\times\mathbb{Z}_{16}$,{(1,3),(1,4),(0,4)})

H_0:(0,0),(1,12),(1,0),(0,12),(0,8),(1,5),(0,2),(1,6),(0,3),(1,15),(1,3),(0,15),(1,2),(1,14),(0,11),(1,8),(1,4),(0,7),(1,10),(0,6),(1,9),(0,5),(0,1),(0,13),(0,9),(1,13),(0,10),(1,7),(1,11),(0,14),(1,1),(0,4)

H_1:(1,6),(1,2),(0,6),(1,3),(1,7),(0,3),(1,0),(0,4),(0,8),(1,4),(0,1),(1,5),(1,1),(0,13),(1,9),(1,13),(0,0),(0,12),(1,8),

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Table 1.6 – continued from previous page

(0,5),(0,9),(1,12),(0,15),(1,11),(0,7),(0,11),(1,15),(0,2),(1,14),(0,10),(0,14),(1,10)
H_2:(1,5),(0,9),(1,6),(0,10),(0,6),(0,2),(0,14),(1,2),(0,5),(1,1),(1,13),(0,1),(1,14),(1,10),(0,13),(1,0),(1,4),(0,0),(1,3), (0,7),(0,3),(0,15),(0,11),(1,7),(0,4),(1,8),(1,12),(0,8),(1,11),(1,15),(0,12),(1,9)
$\textsc{Cay}(\mathbb{Z}_2\times\mathbb{Z}_{16},\{(1,3),(0,5),(1,1)\})$
H_0:(1,6),(0,3),(0,8),(1,11),(0,12),(0,1),(0,6),(1,3),(0,0),(0,11),(1,12),(0,9),(0,4),(1,5),(1,0),(0,15),(0,10),(1,7),(1,2), (0,5),(1,4),(1,9),(1,14),(0,13),(0,2),(1,15),(1,10),(0,7),(1,8),(1,13),(0,14),(1,1)
H_1:(0,10),(0,5),(1,6),(0,7),(0,2),(1,5),(0,8),(1,7),(0,4),(1,3),(1,8),(0,9),(1,10),(0,11),(0,6),(1,9),(0,12),(1,13),(0,0), (1,1),(1,12),(0,13),(1,0),(0,3),(1,2),(0,15),(1,14),(0,1),(1,4),(1,15),(0,14),(1,11)
H_2:(1,15),(0,0),(0,5),(1,8),(0,11),(1,14),(1,3),(0,2),(1,1),(0,4),(0,15),(1,12),(1,7),(0,6),(1,5),(1,10),(0,13),(0,8),(1,9), (0,10),(1,13),(1,2),(0,1),(1,0),(1,11),(1,6),(0,9),(0,14),(0,3),(1,4),(0,7),(0,12)
$\textsc{Cay}(\mathbb{Z}_2\times\mathbb{Z}_{16},\{(0,2),(1,6),(0,7)\})$
H_0:(0,6),(1,0),(1,2),(1,9),(1,11),(1,13),(0,3),(0,1),(0,10),(1,4),(1,6),(1,8),(0,2),(0,0),(1,10),(0,4),(1,14),(1,12),(1,3), (0,9),(1,15),(0,5),(0,12),(0,14),(0,7),(1,1),(0,11),(0,13),(1,7),(1,5),(0,15),(0,8)
H_1:(1,9),(1,7),(1,0),(0,10),(0,8),(1,14),(1,5),(0,11),(0,2),(0,4),(0,6),(1,12),(1,10),(1,1),(1,3),(0,13),(0,15),(0,1),(1,11), (0,5),(0,14),(1,8),(1,15),(1,6),(0,0),(0,9),(0,7),(1,13),(1,4),(1,2),(0,12),(0,3)
H_2:(1,12),(0,2),(0,9),(0,11),(0,4),(0,13),(0,6),(0,15),(1,9),(1,0),(1,14),(1,7),(0,1),(0,8),(1,2),(1,11),(1,4),(0,14),(0,0), (0,7),(0,5),(0,3),(0,10),(0,12),(1,6),(1,13),(1,15),(1,1),(1,8),(1,10),(1,3),(1,5)
$\textsc{Cay}(\mathbb{Z}_2\times\mathbb{Z}_{16},\{(0,1),(1,5),(0,6)\})$
H_0:(1,2),(1,1),(1,7),(0,12),(0,11),(1,0),(1,6),(1,5),(0,10),(0,4),(0,5),(0,6),(0,0),(0,1),(0,2),(0,8),(0,9),(0,3),(0,13), (1,8),(1,9),(1,15),(1,14),(1,13),(1,3),(0,14),(0,15),(1,4),(1,10),(1,11),(1,12),(0,7)
H_1:(0,3),(1,8),(1,2),(1,12),(0,1),(0,7),(0,8),(0,14),(0,13),(0,12),(1,1),(0,6),(1,11),(1,5),(0,0),(0,10),(1,15),(0,4),(1,9), (1,3),(1,4),(1,14),(0,9),(0,15),(1,10),(1,0),(0,5),(0,11),(1,6),(1,7),(1,13),(0,2)
H_2:(1,11),(0,0),(0,15),(0,5),(1,10),(1,9),(0,14),(0,4),(0,3),(1,14),(1,8),(1,7),(0,2),(0,12),(0,6),(0,7),(0,13),(1,2),(1,3), (0,8),(1,13),(1,12),(1,6),(0,1),(0,11),(0,10),(0,9),(1,4),(1,5),(1,15),(1,0),(1,1)
$\textsc{Cay}(\mathbb{Z}_2\times\mathbb{Z}_{16},\{(1,3),(1,4),(0,5)\})$
H_0:(0,12),(1,0),(0,3),(0,14),(0,9),(1,5),(1,10),(0,6),(1,2),(0,5),(0,0),(1,13),(0,1),(1,4),(1,9),(0,13),(1,1),(1,6),(1,11), (0,8),(1,12),(0,15),(1,3),(1,8),(0,4),(1,7),(0,10),(1,14),(0,11),(1,15),(0,2),(0,7)
H_1:(0,0),(1,3),(1,14),(0,1),(1,5),(0,2),(0,13),(1,0),(1,11),(0,14),(1,10),(0,7),(1,4),(0,8),(0,3),(1,15),(0,12),(1,8),(0,5), (1,9),(0,6),(0,11),(1,7),(1,2),(1,13),(0,9),(1,6),(0,10),(0,15),(0,4),(1,1),(1,12)
H_2:(0,7),(1,3),(0,6),(0,1),(0,12),(1,9),(1,14),(0,2),(1,6),(0,3),(1,7),(1,12),(0,9),(0,4),(1,0),(1,5),(0,8),(0,13),(1,10), (1,15),(1,4),(0,0),(0,11),(1,8),(1,13),(0,10),(0,5),(1,1),(0,14),(1,2),(0,15),(1,11)
$\textsc{Cay}(\mathbb{Z}_2\times\mathbb{Z}_{16},\{(0,1),(1,5),(1,6)\})$
H_0:(0,7),(0,8),(1,3),(0,13),(0,12),(1,2),(1,1),(0,11),(1,0),(0,6),(0,5),(0,4),(1,10),(1,11),(0,1),(1,6),(1,5),(0,0),(0,15), (1,4),(0,9),(0,10),(1,15),(1,14),(0,3),(1,9),(0,14),(1,8),(1,7),(0,2),(1,12),(1,13)
H_1:(0,10),(1,4),(0,14),(0,13),(1,2),(1,3),(0,9),(1,14),(0,8),(1,13),(0,3),(0,4),(1,9),(1,8),(0,2),(0,1),(0,0),(1,10),(0,15), (1,5),(0,11),(1,6),(1,7),(0,12),(1,1),(0,6),(0,7),(1,12),(1,11),(0,5),(1,15),(1,0)
H_2:(0,8),(1,2),(0,7),(1,1),(1,0),(0,5),(1,10),(1,9),(0,15),(0,14),(1,3),(1,4),(1,5),(0,10),(0,11),(0,12),(1,6),(0,0),(1,11), (0,6),(1,12),(0,1),(1,7),(0,13),(1,8),(0,3),(0,2),(1,13),(1,14),(0,4),(1,15),(0,9)
$\textsc{Cay}(\mathbb{Z}_2\times\mathbb{Z}_{16},\{(1,4),(1,5),(1,7)\})$
H_0:(1,1),(0,8),(1,12),(0,3),(1,7),(0,11),(1,0),(0,4),(1,11),(0,0),(1,4),(0,13),(1,2),(0,14),(1,5),(0,1),(1,6),(0,2),(1,9), (0,5),(1,10),(0,15),(1,8),(0,12),(1,3),(0,7),(1,14),(0,9),(1,13),(0,6),(1,15),(0,10)
H_1:(1,1),(0,12),(1,0),(0,7),(1,2),(0,6),(1,10),(0,14),(1,3),(0,10),(1,5),(0,9),(1,4),(0,15),(1,11),(0,2),(1,7),(0,0),(1,9), (0,4),(1,8),(0,3),(1,14),(0,5),(1,12),(0,1),(1,13),(0,8),(1,15),(0,11),(1,6),(0,13)
H_2:(1,0),(0,9),(1,2),(0,11),(1,4),(0,8),(1,3),(0,15),(1,6),(0,10),(1,14),(0,2),(1,13),(0,4),(1,15),(0,3),(1,10),(0,1),(1,8), (0,13),(1,9),(0,14),(1,7),(0,12),(1,5),(0,0),(1,12),(0,7),(1,11),(0,6),(1,1),(0,5)
$\textsc{Cay}(\mathbb{Z}_2\times\mathbb{Z}_{16},\{(1,3),(1,4),(1,5)\})$
H_0:(1,6),(0,1),(1,4),(0,8),(1,11),(0,0),(1,3),(0,6),(1,9),(0,4),(1,1),(0,14),(1,2),(0,15),(1,12),(0,7),(1,10),(0,5),(1,0), (0,13),(1,8),(0,11),(1,7),(0,12),(1,15),(0,10),(1,5),(0,2),(1,13),(0,9),(1,14),(0,3)
H_1:(0,9),(1,5),(0,1),(1,12),(0,0),(1,4),(0,15),(1,3),(0,8),(1,13),(0,10),(1,7),(0,4),(1,0),(0,12),(1,9),(0,14),(1,10),(0,6), (1,11),(0,7),(1,2),(0,13),(1,1),(0,5),(1,8),(0,3),(1,15),(0,2),(1,14),(0,11),(1,6)
H_2:(1,0),(0,3),(1,7),(0,2),(1,6),(0,10),(1,14),(0,1),(1,13),(0,0),(1,5),(0,8),(1,12),(0,9),(1,4),(0,7),(1,3),(0,14),(1,11),

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Table 1.6 – continued from previous page

(0,15),(1,10),(0,13),(1,9),(0,5),(1,2),(0,6),(1,1),(0,12),(1,8),(0,4),(1,15),(0,11)
$\mathrm{CAY}(\mathbb{Z}_2\times\mathbb{Z}_{16},\{(1,6),(0,6),(1,1)\})$
H_0:(1,9),(0,3),(1,2),(0,12),(0,2),(1,3),(0,4),(0,14),(1,13),(0,7),(1,1),(1,7),(0,13),(1,12),(1,6),(1,0),(0,6),(0,0),(0,10),(1,4),(1,10),(0,11),(0,1),(1,11),(0,5),(0,15),(1,5),(1,15),(0,9),(1,8),(1,14),(0,8)
H_1:(0,7),(1,6),(0,0),(1,1),(0,11),(0,5),(1,4),(0,3),(0,9),(0,15),(1,14),(0,4),(1,10),(1,0),(0,1),(1,7),(0,8),(0,2),(1,12),(1,2),(1,8),(0,14),(1,15),(1,9),(0,10),(1,11),(1,5),(0,6),(0,12),(1,13),(1,3),(0,13)
H_2:(1,1),(0,2),(1,8),(0,7),(0,1),(1,2),(0,8),(0,14),(1,4),(1,14),(0,13),(0,3),(1,13),(1,7),(0,6),(1,12),(0,11),(1,5),(0,4),(0,10),(1,0),(0,15),(1,9),(1,3),(0,9),(1,10),(0,0),(1,15),(0,5),(1,6),(0,12),(1,11)
$\mathrm{CAY}(\mathbb{Z}_2\times\mathbb{Z}_{16},\{(1,3),(0,1),(0,4)\})$
H_0:(1,5),(0,8),(0,7),(1,4),(0,1),(0,5),(1,8),(1,7),(1,3),(0,0),(0,4),(0,3),(0,15),(1,12),(1,0),(1,15),(0,2),(0,6),(1,9),(1,10),(1,11),(0,14),(1,1),(1,2),(1,14),(1,13),(0,10),(0,11),(0,12),(0,13),(0,9),(1,6)
H_1:(0,2),(1,5),(1,4),(1,0),(1,1),(0,4),(0,8),(0,9),(0,5),(1,2),(1,6),(1,10),(0,13),(0,1),(1,14),(0,11),(0,15),(0,14),(0,10),(1,7),(1,11),(1,12),(1,8),(1,9),(1,13),(0,0),(0,12),(1,15),(1,3),(0,6),(0,7),(0,3)
H_2:(1,7),(0,4),(0,5),(0,6),(0,10),(0,9),(1,12),(1,13),(1,1),(1,5),(1,9),(0,12),(0,8),(1,11),(1,15),(1,14),(1,10),(0,7),(0,11),(1,8),(1,4),(1,3),(1,2),(0,15),(0,0),(0,1),(0,2),(0,14),(0,13),(1,0),(0,3),(1,6)
$\mathrm{CAY}(\mathbb{Z}_2\times\mathbb{Z}_{16},\{(0,1),(1,4),(0,2)\})$
H_0:(1,1),(1,15),(1,0),(1,2),(1,3),(0,7),(0,6),(0,4),(0,2),(0,0),(1,12),(1,13),(1,14),(0,10),(0,9),(0,8),(1,4),(1,5),(1,7),(1,6),(1,8),(0,12),(0,11),(0,13),(0,14),(1,10),(1,9),(1,11),(0,15),(0,1),(0,3),(0,5)
H_1:(0,1),(1,5),(1,3),(1,1),(1,2),(1,4),(0,0),(0,14),(0,15),(0,13),(1,9),(0,5),(0,4),(1,8),(1,7),(0,11),(0,9),(0,7),(0,8),(0,6),(1,10),(1,11),(1,12),(1,14),(1,0),(0,12),(0,10),(1,6),(0,2),(0,3),(1,15),(1,13)
H_2:(0,4),(1,0),(1,1),(0,13),(0,12),(0,14),(1,2),(0,6),(0,5),(0,7),(1,11),(1,13),(0,9),(1,5),(1,6),(1,4),(1,3),(0,15),(0,0),(0,1),(0,2),(1,14),(1,15),(0,11),(0,10),(0,8),(1,12),(1,10),(1,8),(1,9),(1,7),(0,3)

Table 1.7: Hamilton decompositions for Cayley graphs on $\mathbb{Z}_4\times\mathbb{Z}_8$.

$\mathrm{CAY}(\mathbb{Z}_4\times\mathbb{Z}_8,\{(3,1),(3,2),(2,3)\})$
H_0:(1,1),(0,3),(2,0),(3,7),(1,4),(3,1),(1,6),(0,0),(3,2),(2,4),(3,3),(0,1),(1,7),(3,4),(2,5),(0,2),(2,7),(1,0),(2,6),(3,5),(1,2),(2,1),(3,0),(0,6),(2,3),(1,5),(0,7),(2,2),(0,5),(1,3),(0,4),(3,6)
H_1:(0,3),(3,4),(1,1),(0,2),(1,0),(0,1),(3,2),(2,3),(0,0),(3,1),(0,7),(3,0),(2,2),(1,4),(0,5),(2,0),(1,2),(0,4),(3,5),(2,7),(3,6),(1,3),(2,1),(3,7),(0,6),(1,5),(2,4),(1,6),(3,3),(2,5),(1,7),(2,6)
H_2:(3,3),(1,0),(3,5),(0,3),(1,2),(3,7),(0,5),(3,6),(2,0),(1,1),(2,7),(0,4),(2,1),(0,6),(1,4),(2,3),(3,1),(2,2),(1,3),(3,0),(1,5),(3,2),(1,7),(0,0),(2,5),(1,6),(0,7),(2,4),(0,1),(2,6),(3,4),(0,2)
$\mathrm{CAY}(\mathbb{Z}_4\times\mathbb{Z}_8,\{(3,1),(2,1),(2,2)\})$
H_0:(1,1),(0,3),(2,0),(3,7),(1,4),(3,1),(1,6),(0,0),(3,2),(2,4),(3,3),(0,1),(1,7),(3,4),(2,5),(0,2),(2,7),(1,0),(2,6),(3,5),(1,2),(2,1),(3,0),(0,6),(2,3),(1,5),(0,7),(2,2),(0,5),(1,3),(0,4),(3,6)
H_1:(0,3),(3,4),(1,1),(0,2),(1,0),(0,1),(3,2),(2,3),(0,0),(3,1),(0,7),(3,0),(2,2),(1,4),(0,5),(2,0),(1,2),(0,4),(3,5),(2,7),(3,6),(1,3),(2,1),(3,7),(0,6),(1,5),(2,4),(1,6),(3,3),(2,5),(1,7),(2,6)
H_2:(3,3),(1,0),(3,5),(0,3),(1,2),(3,7),(0,5),(3,6),(2,0),(1,1),(2,7),(0,4),(2,1),(0,6),(1,4),(2,3),(3,1),(2,2),(1,3),(3,0),(1,5),(3,2),(1,7),(0,0),(2,5),(1,6),(0,7),(2,4),(0,1),(2,6),(3,4),(0,2)
$\mathrm{CAY}(\mathbb{Z}_4\times\mathbb{Z}_8,\{(3,1),(1,1),(1,2)\})$
H_0:(1,1),(0,3),(2,0),(3,7),(1,4),(3,1),(1,6),(0,0),(3,2),(2,4),(3,3),(0,1),(1,7),(3,4),(2,5),(0,2),(2,7),(1,0),(2,6),(3,5),(1,2),(2,1),(3,0),(0,6),(2,3),(1,5),(0,7),(2,2),(0,5),(1,3),(0,4),(3,6)
H_1:(0,3),(3,4),(1,1),(0,2),(1,0),(0,1),(3,2),(2,3),(0,0),(3,1),(0,7),(3,0),(2,2),(1,4),(0,5),(2,0),(1,2),(0,4),(3,5),(2,7),(3,6),(1,3),(2,1),(3,7),(0,6),(1,5),(2,4),(1,6),(3,3),(2,5),(1,7),(2,6)
H_2:(3,3),(1,0),(3,5),(0,3),(1,2),(3,7),(0,5),(3,6),(2,0),(1,1),(2,7),(0,4),(2,1),(0,6),(1,4),(2,3),(3,1),(2,2),(1,3),(3,0),(1,5),(3,2),(1,7),(0,0),(2,5),(1,6),(0,7),(2,4),(0,1),(2,6),(3,4),(0,2)
$\mathrm{CAY}(\mathbb{Z}_4\times\mathbb{Z}_8,\{(1,4),(0,2),(2,3)\})$
H_0:(1,1),(0,3),(2,0),(3,7),(1,4),(3,1),(1,6),(0,0),(3,2),(2,4),(3,3),(0,1),(1,7),(3,4),(2,5),(0,2),(2,7),(1,0),(2,6),(3,5),(1,2),(2,1),(3,0),(0,6),(2,3),(1,5),(0,7),(2,2),(0,5),(1,3),(0,4),(3,6)
H_1:(0,3),(3,4),(1,1),(0,2),(1,0),(0,1),(3,2),(2,3),(0,0),(3,1),(0,7),(3,0),(2,2),(1,4),(0,5),(2,0),(1,2),(0,4),(3,5),(2,7),(3,6),

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Table 1.7 – continued from previous page

(1,3),(2,1),(3,7),(0,6),(1,5),(2,4),(1,6),(3,3),(2,5),(1,7),(2,6)
H_2:(3,3),(1,0),(3,5),(0,3),(1,2),(3,7),(0,5),(3,6),(2,0),(1,1),(2,7),(0,4),(2,1),(0,6),(1,4),(2,3),(3,1),(2,2),(1,3),(3,0),(1,5),(3,2),(1,7),(0,0),(2,5),(1,6),(0,7),(2,4),(0,1),(2,6),(3,4),(0,2)
CAY($\mathbb{Z}_4\times\mathbb{Z}_8$,{(0,1),(2,2),(1,0)})
H_0:(1,1),(0,3),(2,0),(3,7),(1,4),(3,1),(1,6),(0,0),(3,2),(2,4),(3,3),(0,1),(1,7),(3,4),(2,5),(0,2),(2,7),(1,0),(2,6),(3,5),(1,2),(2,1),(3,0),(0,6),(2,3),(1,5),(0,7),(2,2),(0,5),(1,3),(0,4),(3,6)
H_1:(0,3),(3,4),(1,1),(0,2),(1,0),(0,1),(3,2),(2,3),(0,0),(3,1),(0,7),(3,0),(2,2),(1,4),(0,5),(2,0),(1,2),(0,4),(3,5),(2,7),(3,6),(1,3),(2,1),(3,7),(0,6),(1,5),(2,4),(1,6),(3,3),(2,5),(1,7),(2,6)
H_2:(3,3),(1,0),(3,5),(0,3),(1,2),(3,7),(0,5),(3,6),(2,0),(1,1),(2,7),(0,4),(2,1),(0,6),(1,4),(2,3),(3,1),(2,2),(1,3),(3,0),(1,5),(3,2),(1,7),(0,0),(2,5),(1,6),(0,7),(2,4),(0,1),(2,6),(3,4),(0,2)
CAY($\mathbb{Z}_4\times\mathbb{Z}_8$,{(3,1),(1,0),(2,3)})
H_0:(1,1),(0,3),(2,0),(3,7),(1,4),(3,1),(1,6),(0,0),(3,2),(2,4),(3,3),(0,1),(1,7),(3,4),(2,5),(0,2),(2,7),(1,0),(2,6),(3,5),(1,2),(2,1),(3,0),(0,6),(2,3),(1,5),(0,7),(2,2),(0,5),(1,3),(0,4),(3,6)
H_1:(0,3),(3,4),(1,1),(0,2),(1,0),(0,1),(3,2),(2,3),(0,0),(3,1),(0,7),(3,0),(2,2),(1,4),(0,5),(2,0),(1,2),(0,4),(3,5),(2,7),(3,6),(1,3),(2,1),(3,7),(0,6),(1,5),(2,4),(1,6),(3,3),(2,5),(1,7),(2,6)
H_2:(3,3),(1,0),(3,5),(0,3),(1,2),(3,7),(0,5),(3,6),(2,0),(1,1),(2,7),(0,4),(2,1),(0,6),(1,4),(2,3),(3,1),(2,2),(1,3),(3,0),(1,5),(3,2),(1,7),(0,0),(2,5),(1,6),(0,7),(2,4),(0,1),(2,6),(3,4),(0,2)
CAY($\mathbb{Z}_4\times\mathbb{Z}_8$,{(0,1),(1,4),(1,2)})
H_0:(1,1),(0,3),(2,0),(3,7),(1,4),(3,1),(1,6),(0,0),(3,2),(2,4),(3,3),(0,1),(1,7),(3,4),(2,5),(0,2),(2,7),(1,0),(2,6),(3,5),(1,2),(2,1),(3,0),(0,6),(2,3),(1,5),(0,7),(2,2),(0,5),(1,3),(0,4),(3,6)
H_1:(0,3),(3,4),(1,1),(0,2),(1,0),(0,1),(3,2),(2,3),(0,0),(3,1),(0,7),(3,0),(2,2),(1,4),(0,5),(2,0),(1,2),(0,4),(3,5),(2,7),(3,6),(1,3),(2,1),(3,7),(0,6),(1,5),(2,4),(1,6),(3,3),(2,5),(1,7),(2,6)
H_2:(3,3),(1,0),(3,5),(0,3),(1,2),(3,7),(0,5),(3,6),(2,0),(1,1),(2,7),(0,4),(2,1),(0,6),(1,4),(2,3),(3,1),(2,2),(1,3),(3,0),(1,5),(3,2),(1,7),(0,0),(2,5),(1,6),(0,7),(2,4),(0,1),(2,6),(3,4),(0,2)
CAY($\mathbb{Z}_4\times\mathbb{Z}_8$,{(1,3),(3,3),(2,1)})
H_0:(1,1),(0,3),(2,0),(3,7),(1,4),(3,1),(1,6),(0,0),(3,2),(2,4),(3,3),(0,1),(1,7),(3,4),(2,5),(0,2),(2,7),(1,0),(2,6),(3,5),(1,2),(2,1),(3,0),(0,6),(2,3),(1,5),(0,7),(2,2),(0,5),(1,3),(0,4),(3,6)
H_1:(0,3),(3,4),(1,1),(0,2),(1,0),(0,1),(3,2),(2,3),(0,0),(3,1),(0,7),(3,0),(2,2),(1,4),(0,5),(2,0),(1,2),(0,4),(3,5),(2,7),(3,6),(1,3),(2,1),(3,7),(0,6),(1,5),(2,4),(1,6),(3,3),(2,5),(1,7),(2,6)
H_2:(3,3),(1,0),(3,5),(0,3),(1,2),(3,7),(0,5),(3,6),(2,0),(1,1),(2,7),(0,4),(2,1),(0,6),(1,4),(2,3),(3,1),(2,2),(1,3),(3,0),(1,5),(3,2),(1,7),(0,0),(2,5),(1,6),(0,7),(2,4),(0,1),(2,6),(3,4),(0,2)
CAY($\mathbb{Z}_4\times\mathbb{Z}_8$,{(1,4),(3,3),(1,0)})
H_0:(1,1),(0,3),(2,0),(3,7),(1,4),(3,1),(1,6),(0,0),(3,2),(2,4),(3,3),(0,1),(1,7),(3,4),(2,5),(0,2),(2,7),(1,0),(2,6),(3,5),(1,2),(2,1),(3,0),(0,6),(2,3),(1,5),(0,7),(2,2),(0,5),(1,3),(0,4),(3,6)
H_1:(0,3),(3,4),(1,1),(0,2),(1,0),(0,1),(3,2),(2,3),(0,0),(3,1),(0,7),(3,0),(2,2),(1,4),(0,5),(2,0),(1,2),(0,4),(3,5),(2,7),(3,6),(1,3),(2,1),(3,7),(0,6),(1,5),(2,4),(1,6),(3,3),(2,5),(1,7),(2,6)
H_2:(3,3),(1,0),(3,5),(0,3),(1,2),(3,7),(0,5),(3,6),(2,0),(1,1),(2,7),(0,4),(2,1),(0,6),(1,4),(2,3),(3,1),(2,2),(1,3),(3,0),(1,5),3,2),(1,7),(0,0),(2,5),(1,6),(0,7),(2,4),(0,1),(2,6),(3,4),(0,2)
CAY($\mathbb{Z}_4\times\mathbb{Z}_8$,{(3,2),(2,1),(2,3)})
H_0:(1,1),(0,3),(2,0),(3,7),(1,4),(3,1),(1,6),(0,0),(3,2),(2,4),(3,3),(0,1),(1,7),(3,4),(2,5),(0,2),(2,7),(1,0),(2,6),(3,5),(1,2),(2,1),(3,0),(0,6),(2,3),(1,5),(0,7),(2,2),(0,5),(1,3),(0,4),(3,6)
H_1:(0,3),(3,4),(1,1),(0,2),(1,0),(0,1),(3,2),(2,3),(0,0),(3,1),(0,7),(3,0),(2,2),(1,4),(0,5),(2,0),(1,2),(0,4),(3,5),(2,7),(3,6),(1,3),(2,1),(3,7),(0,6),(1,5),(2,4),(1,6),(3,3),(2,5),(1,7),(2,6)
H_2:(3,3),(1,0),(3,5),(0,3),(1,2),(3,7),(0,5),(3,6),(2,0),(1,1),(2,7),(0,4),(2,1),(0,6),(1,4),(2,3),(3,1),(2,2),(1,3),(3,0),(1,5),(3,2),(1,7),(0,0),(2,5),(1,6),(0,7),(2,4),(0,1),(2,6),(3,4),(0,2)
CAY($\mathbb{Z}_4\times\mathbb{Z}_8$,{(0,1),(1,4),(2,1)})
H_0:(1,1),(0,3),(2,0),(3,7),(1,4),(3,1),(1,6),(0,0),(3,2),(2,4),(3,3),(0,1),(1,7),(3,4),(2,5),(0,2),(2,7),(1,0),(2,6),(3,5),(1,2),(2,1),(3,0),(0,6),(2,3),(1,5),(0,7),(2,2),(0,5),(1,3),(0,4),(3,6)
H_1:(0,3),(3,4),(1,1),(0,2),(1,0),(0,1),(3,2),(2,3),(0,0),(3,1),(0,7),(3,0),(2,2),(1,4),(0,5),(2,0),(1,2),(0,4),(3,5),(2,7),(3,6),(1,3),(2,1),(3,7),(0,6),(1,5),(2,4),(1,6),(3,3),(2,5),(1,7),(2,6)

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H_2:(3,3),(1,0),(3,5),(0,3),(1,2),(3,7),(0,5),(3,6),(2,0),(1,1),(2,7),(0,4),(2,1),(0,6),(1,4),(2,3),(3,1),(2,2),(1,3),(3,0),(1,5),(3,2),(1,7),(0,0),(2,5),(1,6),(0,7),(2,4),(0,1),(2,6),(3,4),(0,2)

Table 1.8: Hamilton decompositions for Cayley graphs on $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_8$.

CAY($\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_8$,{(1,0,3),(0,0,3),(0,1,2)})
H_0:(0,1,0),(0,0,6),(0,0,3),(0,0,0),(0,0,5),(0,0,2),(0,0,7),(0,1,1),(0,1,6),(0,1,3),(1,1,6),(1,0,4),(1,0,7),(0,0,4),(0,0,1),(0,1,7),(1,1,2),(0,1,5),(1,1,0),(1,1,3),(1,0,1),(1,0,6),(1,1,4),(1,0,2),(1,0,5),(1,0,0),(1,0,3),(1,1,1),(0,1,4),(1,1,7),(0,1,2),(1,1,5)
H_1:(0,1,6),(1,1,3),(0,1,0),(0,0,2),(0,1,4),(0,0,6),(0,0,1),(0,1,3),(0,0,5),(1,0,0),(0,0,3),(0,1,5),(0,0,7),(1,0,4),(1,0,1),(0,0,4),(0,1,2),(0,1,7),(1,1,4),(0,1,1),(1,1,6),(1,1,1),(1,0,7),(1,0,2),(1,1,0),(1,0,6),(1,0,3),(1,1,5),(1,1,2),(1,1,7),(1,0,5),(0,0,0)
H_2:(1,1,2),(1,0,0),(1,1,6),(1,1,3),(1,0,5),(0,0,2),(1,0,7),(1,1,5),(1,1,0),(0,1,3),(0,1,0),(0,1,5),(0,1,2),(0,0,0),(1,0,3),(0,0,6),(1,0,1),(1,1,7),(1,1,4),(1,1,1),(0,1,6),(0,0,4),(0,0,7),(1,0,2),(0,0,5),(0,1,7),(0,1,4),(0,1,1),(0,0,3),(1,0,6),(0,0,1),(1,0,4)
CAY($\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_8$,{(0,0,1),(1,0,3),(1,1,1)})
H_0:(1,1,7),(0,0,6),(0,0,7),(0,0,0),(1,0,3),(0,1,2),(0,1,3),(1,0,2),(1,0,1),(0,1,0),(0,1,7),(1,0,0),(0,1,1),(1,1,4),(0,0,5),(1,1,6),(1,1,5),(0,0,4),(0,0,3),(0,0,2),(0,0,1),(1,1,0),(0,1,5),(1,0,4),(1,0,5),(1,0,6),(1,0,7),(0,1,6),(1,1,3),(1,1,2),(1,1,1),(0,1,4)
H_1:(0,0,2),(1,0,7),(0,1,0),(1,1,3),(0,0,4),(0,0,5),(0,0,6),(1,0,3),(1,0,2),(0,1,1),(1,1,6),(1,1,7),(1,1,0),(0,0,7),(1,0,4),(0,1,3),(0,1,4),(0,1,5),(1,0,6),(0,0,1),(1,1,2),(0,0,3),(1,0,0),(1,0,1),(0,1,2),(1,1,5),(1,1,4),(0,1,7),(0,1,6),(1,0,5),(0,0,0),(1,1,1)
H_2:(1,0,3),(0,1,4),(1,0,5),(0,0,2),(1,1,3),(1,1,4),(0,0,3),(1,0,6),(0,1,7),(1,1,2),(0,1,5),(0,1,6),(1,1,1),(1,1,0),(0,1,3),(1,1,6),(0,0,7),(1,0,2),(0,0,5),(1,0,0),(1,0,7),(0,0,4),(1,0,1),(0,0,6),(1,1,5),(0,1,0),(0,1,1),(0,1,2),(1,1,7),(0,0,0),(0,0,1),(1,0,4)
CAY($\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_8$,{(1,0,2),(0,1,3),(1,1,2)})
H_0:(1,0,2),(0,0,0),(0,1,3),(1,0,1),(0,0,3),(0,1,6),(1,0,0),(0,0,6),(1,0,4),(0,0,2),(0,1,7),(0,0,4),(1,1,2),(1,0,5),(1,1,0),(1,0,3),(0,1,5),(1,0,7),(1,1,4),(0,1,2),(0,0,5),(1,1,7),(0,1,1),(1,1,3),(0,0,1),(0,1,4),(1,1,6),(0,1,0),(1,0,6),(1,1,1),(0,0,7),(1,1,5)
H_1:(0,1,0),(0,0,3),(1,1,1),(0,1,3),(1,1,5),(0,1,7),(1,0,5),(0,0,7),(0,1,2),(1,0,4),(0,1,6),(1,1,0),(0,0,6),(0,1,1),(1,0,3),(0,0,1),(1,0,7),(0,0,5),(1,1,3),(1,0,0),(0,0,2),(1,1,4),(1,0,1),(1,1,6),(0,0,4),(1,0,6),(0,1,4),(1,0,2),(1,1,7),(0,1,5),(0,0,0),(1,1,2)
H_2:(1,0,3),(0,0,5),(0,1,0),(1,0,2),(0,0,4),(0,1,1),(1,0,7),(1,1,2),(0,1,4),(0,0,7),(1,0,1),(0,1,7),(1,1,1),(1,0,4),(1,1,7),(0,0,1),(0,1,6),(1,1,4),(0,0,6),(0,1,3),(1,0,5),(0,0,3),(1,1,5),(1,0,0),(0,1,2),(1,1,0),(0,0,2),(0,1,5),(1,1,3),(1,0,6),(0,0,0),(1,1,6)

Appendix B

Source code

2.1 MAGMA code

The MAGMA function, `CayBuild3(A,s1,s2,s3)`, constructs a Cayley graph of A relative to connection set $S = \{s_1, s_2, s_3\}$. The function, `CGraphs3(A,Fout)`, outputs to file all generating sets $S = \{s_1, s_2, s_3\}$ for an abelian group A, where $2 < |s_i| < |A|$, for $i = 1, 2, 3$, such that the corresponding set of 6-regular, connected, Cayley graphs are pairwise non-isomorphic.

```
>intrinsic CayBuild3(A::GrpAb, s1::GrpAbElt, s2::GrpAbElt, s3::GrpAbElt ) ->
>  GrphUnd,GrphVertSet,GrphEdgeSet
>{Constructs the Cayley Graph of A generated by s1, s2, and s3.}
>  V:={a: a in A};
>  E:={};
>  for x in A do
>    e := {x, x*s1 };
>    f := {x, x*s2 };
>    g := {x, x*s3 };
>    Include(~E,e);
>    Include(~E,f);
>    Include(~E,g);
>  end for;
>    G,V,E := Graph< V | E >;
>  return G,V,E;
>end intrinsic

>intrinsic CGraphs3(A::GrpAb, F::File)
>{Prints all Three-Element generating sets of A.}
>  V := {a: a in A | Order(a) ne 2 and Order(a) ne 1 and Order(a) ne Order(A)};
>  for a in A do
>    if a in V then
>       Exclude(~V, -a);
>    end if;
>  end for;
>  S := Subsets(V,3);
>  SI := {};
>  for s in S do
```

```
>    H := sub<A|s>;
>    if H eq A then
>       t := SetToIndexedSet(s);
>       Include(~SI, t);
>    end if;
>  end for;
>  SJ := SetToIndexedSet(SI);
>  TJ := IndexedSetToSet(SJ);
>  n := #SJ;
>  for i := 1 to n-1 do
>     G1,E1,V1 := CayBuild3(A,SJ[i][1], SJ[i][2], SJ[i][3]);
>     for j := i+1 to n do
>       G2,E2,V2 := CayBuild3(A,SJ[j][1], SJ[j][2], SJ[j][3]);
>       if IsIsomorphic(G1,G2) then
>         TJj := IndexedSetToSet(SJ[j]);
>         Exclude(~TJ, TJj);
>       end if;
>     end for;
>  end for;
>  SK := SetToIndexedSet(TJ);
>  m := #SK;
>  fprintf F, "%o %o %o\n", m , Order(A), 3;
>  for i := 1 to m do
>    fprintf F, "%o %o %o\n", SK[i][1],SK[i][2],SK[i][3];
>  end for;
>end intrinsic;
```

2.2 C code

The following C-programs were used to find random Hamilton cycle decompositions for the graphs obtained using MAGMA in Section 2.1. They were written with Donald L. Kreher.

2.2.1 Constructing the Cayley graphs

```
/*-----------------------GetGraph.c-----------------------*/
#include<stdlib.h>
#include<stdio.h>

FILE *f;
int n,m,s;
int *S;

main( int ac, char * av[])
{
  int i,j;
  if(ac!=2)
  {
   fprintf(stderr,"Usage: %s fname\n",av[0]);
   exit(1);
  }
```

```
  if(!(f=fopen(av[1],"r") ) )
  {
   fprintf(stderr,"%s cannot open %s\n",av[1]);
   exit(1);
  }
  fscanf(f," %d %d ",&n,&s);
  S = (int *)calloc(s,sizeof(int));
  for(i=0;i<s;i++) fscanf(f,"%d",&S[i]);
  m=n*s;
  printf("  %d %d\n",n,m);
  for(i=0;i<n;i++)
  {
   for(j=0;j<s;j++) printf(" %d %d\n",i , (n+i+S[j])%n);
  }
}

/*-----------------------GetGraph2.c-----------------------*/
#include<stdlib.h>
#include<stdio.h>

FILE *f;
int n,na,nb,m,s;
int S[10][2];

main( int ac, char * av[])
{
  int i,j;
  int i0,a0,b0;
  int i1,a1,b1;
  if(ac!=2)
  {
   fprintf(stderr,"Usage: %s fname\n",av[0]);
   exit(1);
  }
  if(!(f=fopen(av[1],"r") ) )
  {
   fprintf(stderr,"%s cannot open %s\n",av[1]);
   exit(1);
  }
  fscanf(f," %d %d %d ",&na,&nb,&s);
  for(i=0;i<s;i++) fscanf(f,"%d %d",&S[i][0],&S[i][1]);
  n=na*nb;
  m=n*s;
  printf("  %d %d\n",n,m);
  for(i0=0;i0<n;i0++)
  {
   a0=i0%na;
   b0=(i0-a0)/na;
   for(j=0;j<s;j++)
   {
    a1=(na+a0+S[j][0])%na;
    b1=(nb+b0+S[j][1])%nb;
    i1= a1+na*b1;
    printf(" %d %d\n",i0 , i1);
```

```
   }
  }
}

/*-----------------------GetGraph3.c-----------------------*/
#include<stdlib.h>
#include<stdio.h>

FILE *f;
int n,na,nb,nc,m,g,nt;
int S[10][10][10];
int T[10][3];

main( int ac, char * av[] )
{
  int i,j,k,counter;
  int i0,a0,b0,c0,j0,k0,j1,k1,l;
  int i1,a1,b1,c1;
  if(ac!=2)
  {
   fprintf(stderr,"Usage: %s fname\n",av[0]);
   exit(1);
  }
  if(!(f=fopen(av[1],"r") ) )
  {
   fprintf(stderr,"%s cannot open %s\n",av[1]);
   exit(1);
  }
  fscanf(f," %d %d %d %d ",&na,&nb,&nc,&g);
  for(i=0;i<g;i++)
    fscanf(f,"%d %d %d",&T[i][0],&T[i][1],&T[i][2]);
  n  = na*nb*nc;
  nt = na*nb;
  m  = n*g;
  printf("  %d %d\n",n,m);
  counter = 0;
  for(i0=0;i0<na;i0++)
   for(j0=0;j0<nb;j0++)
    for(k0=0;k0<nc;k0++)
        {
        S[i0][j0][k0] = counter;
        counter++;
        }
  for(i0=0;i0<na;i0++)
   for(j0=0;j0<nb;j0++)
        for(k0=0;k0<nc;k0++)
         for(l=0;l<3;l++)
         {
          i1 = (na+i0 + T[l][0])%na;
          j1 = (nb+j0 + T[l][1])%nb;
          k1 = (nc+k0 + T[l][2])%nc;
          printf(" %d %d\n",S[i0][j0][k0],S[i1][j1][k1]);
         }
}
```

2.2.2 Hamilton cycles via a randomized greedy algorithm

```
/*-----------------------RHC.c-----------------------*/
#include<stdlib.h>
#include<stdio.h>

FILE *f;
int n,m;
int **A;
int **C;
int *X;
int *N;
int L;
int seed;
int *done;
int *R;

/* ---------------genalea----------------*/

double genalea (x0)
     int *x0;
{
 int m = 2147483647;
 int a = 16807 ;
 int b = 127773 ;
 int c = 2836 ;
 int x1, k;

 k = (int) ((*x0)/b) ;
 x1 = a*(*x0 - k*b) - k*c ;
 if(x1 < 0) x1 = x1 + m;
 *x0 = x1;

 if(((double)x1/(double)m > 0.0001) &&
   ((double)x1/(double)m < 0.99999))
  return((double)x1/(double)m);
 else return(genalea(x0));
}

/* ---------------Randomize---------------*/

void Randomize(int h)
{
 int i,j,x,y;
 for(i=0;i<N[h];i++)
   R[i]=N[h]*genalea(&seed);
 for(i=1;i<N[h];i++)
 {
  x=R[i];
  y=C[h][i];
  j=i-1;
  while(j >=0 && R[j]> x )
  {
    R[j+1]=R[j];
```

```
    C[h][j+1]=C[h][j];
    j=j-1;
  }
  R[j+1]=x;
  C[h][j+1]=y;
 }
}

/* ----------------BT------------------*/

void BT(int ell)
{
 int i;
 if(ell==n)
 {
   for(i=0;i<n;i++) printf(" %d",X[i]);
   printf("\n");
 f=fopen("seed","w");
 fprintf(f," %d\n",seed);
 fclose(f);
 exit(1);
 }
 N[ell]=0;
 if(ell==0)
 {
  for(i=0;i<n;i++) C[ell][N[ell]++]=i;
 }
 else if(ell == (n-1) )
 {
  for(i=0;i<n;i++)
    if(A[X[0]][i] && A[X[ell-1]][i] && !done[i])
          C[ell][N[ell]++]=i;
 }
 else
 {
  for(i=0;i<n;i++)
    if(A[X[ell-1]][i] && !done[i])
          C[ell][N[ell]++]=i;
 }
 Randomize(ell);

 for(i=0;i<N[ell];i++)
 {
   X[ell]=C[ell][i];
   done[X[ell]]=1;
   BT(ell+1);
   done[X[ell]]=0;
 }
}

/* ---------------Main---------------*/

main( int ac, char * av[])
{
```

```
  int i,j,x,y;
  setbuf(stdout,0);
  if(ac!=2)
  {
   fprintf(stderr,"Usage: %s fname\n",av[0]);
   exit(1);
  }
  if(!(f=fopen(av[1],"r") ) )
  {
   fprintf(stderr,"%s cannot open %s\n",av[1]);
   exit(1);
  }
  fscanf(f," %d %d ",&n,&m);  //Scans #vertices #edges
  A=(int **)calloc(n,sizeof(int*));
  C=(int **)calloc(n,sizeof(int*));
  N=(int *)calloc(n,sizeof(int));
  X=(int *)calloc(n,sizeof(int));
  done=(int *)calloc(n,sizeof(int));
  R=(int *)calloc(n,sizeof(int));
  for(i=0;i<n;i++) done[i]=0;
  for(i=0;i<n;i++)
  {
   A[i]=(int *)calloc(n,sizeof(int));
   C[i]=(int *)calloc(n,sizeof(int));
   for(j=0;j<n;j++) A[i][j]=0;
  }
  for(i=0;i<m;i++) //Constructs the adjacency matrix, A.
  {
   fscanf(f," %d %d ",&x,&y);
   A[x][y]=1;
   A[y][x]=1;
  }
  fclose(f);

 if((f=fopen("seed","r"))==NULL)
 {
  f=fopen("seed","w");
  printf("Please enter the first seed \n");
  printf("for random number generator:");
  scanf("%d",&seed);
  fprintf(f," %d\n",seed );
  fclose(f);
 }
 f=fopen("seed","r");
 fscanf(f," %d",&seed);
 fclose(f);

 BT(0);
}
```

2.2.3 Obtaining Hamilton decompositions

```
/*------------------------DelHCyc.c------------------------*/
```

```
#include<stdlib.h>
#include<stdio.h>

FILE *f;
int n,m;
int **A;

/*  Deletes the Hamiltonian cycle H from the graph with adjacency matrix A. */

main( int ac, char * av[])
{
  int i,j,x,y,z;
  setbuf(stdout,0);
  if(ac!=3)
  {
   fprintf(stderr,"Usage: %s graph H-cycle\n",av[0]);
   exit(1);
  }
  if(!(f=fopen(av[1],"r") ) )
  {
   fprintf(stderr,"%s cannot open %s\n",av[1]);
   exit(1);
  }
  fscanf(f," %d %d ",&n,&m);
  A=(int **)calloc(n,sizeof(int*));
  for(i=0;i<n;i++)
  {
   A[i]=(int *)calloc(n,sizeof(int));
   for(j=0;j<n;j++) A[i][j]=0;
  }
  for(i=0;i<m;i++)
  {
   fscanf(f," %d %d ",&x,&y);
   A[x][y]=1;
   A[y][x]=1;
  }
  fclose(f);

  if(!(f=fopen(av[2],"r") ) )
  {
   fprintf(stderr,"%s cannot open %s\n",av[2]);
   exit(1);
  }
  fscanf(f," %d ",&x);
  z=x;
  for(i=1;i<n;i++)
  {
   fscanf(f," %d ",&y);
   A[x][y]=0;
   A[y][x]=0;
   x=y;
  }
  y=z;
  A[x][y]=0;
```

```
  A[y][x]=0;
  printf(" %d %d\n",n,m-n);
  for(x=0;x<(n-1);x++)
   for(y=x+1;y<n;y++)
    if(A[x][y]) printf(" %d %d\n",x,y);
}

/*-----------------------GraphLister.c-----------------------*/
#include<stdlib.h>
#include<stdio.h>
int G[20];
int G2[10][2];
int G3[10][3];

main(int ac,char *av[])
{
 int i,j,m,n,g,ng,na,nb,nc ;
 char datafname[20];
 char command[20];
 FILE *FI;
 FILE *FO, *Fout;
 setbuf(stdout,0);
 if(ac!=3)
 {
  fprintf(stderr,"Usage %s fgens numGroups \n",av[0]);
  exit(1);
 }
 if( !(FI=fopen(av[1],"r")) )
 {
   fprintf(stderr,"%s cannot open %s\n", av[0],av[1]);
   exit(1);
 }
 ng = atoi(av[2]);
 if(ng != 1 && ng != 2 && ng!= 3)
 {
   fprintf(stderr,"%s only works with Z_a, Z_a x Z_b, \n");
   fprintf(stderr,"or Z_a x Z_b x Z_c\n", av[0]);
   exit(1);
 }
 if( ng == 1 )          //For Groups Z_n
 {
   fscanf(FI," %d %d %d", &m, &n, &g);
   for(i=0;i<m;i++)
   {
     for(j=0;j<g;j++) fscanf(FI," %d ", &G[j]);
     sprintf(datafname,"D%d",i);
     if( !(FO=fopen(datafname,"w")) )
         {
      fprintf(stderr,"%s cannot open %s\n", av[0],datafname);
      exit(1);
     }
     fprintf(FO," %d %d \n",n,g);
     for(j=0;j<g;j++)
           fprintf(FO,"%d ", G[j]);
```

```
        fprintf(FO,"\n");
    fclose(FO);
    sprintf(command,"./RUN %s %d",datafname,ng);
    system(command);
  }
 }else if( ng == 2 )    // For Groups Z_{na} \times Z_{nb}
 {
  fscanf(FI,"%d %d %d %d", &m, &na, &nb, &g);
  for(i=0;i<m;i++)
  {
    for(j=0;j<g;j++)
          fscanf(FI," %d %d", &G2[j][0], &G2[j][1]);
    sprintf(datafname,"D%d",i);
    if( !(FO=fopen(datafname,"w")) )
        {
     fprintf(stderr,"%s cannot open %s\n", av[0],datafname);
     exit(1);
    }
    fprintf(FO," %d %d %d \n", na,nb,g);
    for(j=0;j<g;j++)
          fprintf(FO,"%d %d  ", G2[j][0], G2[j][1]);
        fprintf(FO,"\n");
    fclose(FO);
    sprintf(command,"./RUN %s %d",datafname,ng);
    system(command);
  }
 }else  //  For Groups Z_{na} \times Z_{nb} \times Z_{nc}
 {
  fscanf(FI,"%d %d %d %d %d", &m, &na, &nb, &nc, &g);
  for(i=0;i<m;i++)
  {
       for(j=0;j<g;j++)
        fscanf(FI," %d %d %d",&G3[j][0],&G3[j][1],&G3[j][2]);
    sprintf(datafname,"D%d",i);
    if( !(FO=fopen(datafname,"w")) )
        {
     fprintf(stderr,"%s cannot open %s\n", av[0],datafname);
     exit(1);
    }
    fprintf(FO," %d %d %d %d \n",na,nb,nc,g);
    for(j=0;j<g;j++)
          fprintf(FO,"%d %d %d ",G3[j][0],G3[j][1],G3[j][2]);
        fprintf(FO,"\n");
    fclose(FO);
    sprintf(command,"./RUN %s %d",datafname,ng);
    system(command);
  }
 }
}
```

2.2.4 Outputting to LaTeX

```
/*-----------------------Convert1.c-----------------------*/
```

```
#include<stdlib.h>
#include<stdio.h>

FILE * Fin, * Fout, * Fgens ;
int * H, ** K;

/* Takes a list, HCYC, of H-decompositions, a list, fgens,
of Cayley Graphs and prints the LaTeX to fout */

main(int ac, char* av[])
{
 int n,i,v,k,j,g;
 if( ac != 4 )
 {
  fprintf(stderr,"Usage %s HCYC fgens fout\n",av[0]);
  exit(1);
 }
 if( !(Fin=fopen(av[1],"r")) )
 {
   fprintf(stderr,"%s cannot open %s\n", av[0],av[1]);
   exit(1);
 }
 if( !(Fgens=fopen(av[2],"r")) )
 {
   fprintf(stderr,"%s cannot open %s\n", av[0],av[2]);
   exit(1);
 }
 if( !(Fout=fopen(av[3],"w")) )
   {
    fprintf(stderr,"%s cannot open %s\n", av[0],av[3]);
    exit(1);
   }
 fscanf(Fgens, "%d ", &n);
 fscanf(Fgens, "%d ", &v);
 fscanf(Fgens, "%d \n", &g);

 H = (int *)calloc(v,sizeof(int));
 K = (int **)calloc(n,sizeof(int*));
 for(i=0;i<n;i++)
 {
   K[i] = (int*)calloc(g,sizeof(int));
   for(j=0;j<g;j++)
     fscanf(Fgens, "%d ", &K[i][j]);
 }
 for(k=0;k<n;k++)
 {
  fprintf(Fout, "$\\cay(\\zed_{%d},\\{%d,",v,K[k][0]);
  for(j=1;j<g-1;j++)
    fprintf(Fout, "%d,",K[k][j]);
  fprintf(Fout, "%d\\}) $ \\\\ \\hline\n",K[k][g-1]);
  for(j=0;j<g;j++)
  {
   fprintf(Fout, "$H_%d: ",j);
   for(i=0;i<v-1;i++)
```

```
   {
    fscanf(Fin, "%d ", &H[i]);
    fprintf(Fout, "%d,", H[i]);
   }
   fscanf(Fin, "%d ", &H[v-1]);
   fprintf(Fout, "%d ", H[v-1]);
   fprintf(Fout, "$ \\\\ \n");
  }
  fprintf(Fout, "\\hline \n");
 }
 free(K);
 free(H);
 close(Fout);
 close(Fin);
 close(Fgens);
}

/*------------------------Convert2.c------------------------*/
#include<stdlib.h>
#include<stdio.h>

FILE *Fin, *Fgens, *Fout;
int * H, **K;
int n,na,nb,m,s,v;
int S[10][2];

main( int ac, char * av[])
{
  int i,j,g,k;
  int i0,a0,b0;
  int i1,a1,b1;
  char L[4];
  if(ac!=4)
  {
   fprintf(stderr,"Usage: %s HCYC fgens fout\n",av[0]);
   exit(1);
  }
 if( !(Fin=fopen(av[1],"r")) )
 {
   fprintf(stderr,"%s cannot open %s\n",av[0],av[1]);
   exit(1);
 }
 if( !(Fgens=fopen(av[2],"r")) )
 {
   fprintf(stderr,"%s cannot open %s\n",av[0],av[2]);
   exit(1);
 }
 if( !(Fout=fopen(av[3],"w")) )
 {
   fprintf(stderr,"%s cannot open %s\n",av[0],av[3]);
   exit(1);
 }
 fscanf(Fgens,"%d %d %d %d\n",&n,&na,&nb,&g);
 v=na*nb;
```

```
 H = (int *)calloc(v,sizeof(int));
 K = (int **)calloc(n,sizeof(int*));

 for(i=0;i<n;i++)
 {
   K[i] = (int*)calloc(2*g,sizeof(int));
   for(j=0;j<2*g;j++)
     fscanf(Fgens, "%d ", &K[i][j]);
 }
 for(k=0;k<n;k++)
 {
  fprintf(Fout,"$\\cay(\\zed_{%d}\\times\\zed_{%d},\\{(%d,%d)",
  na,nb,K[k][0],K[k][1]);
  for(j=2;j<2*g-1;j=j+2)
    fprintf(Fout,",(%d,%d)",K[k][j],K[k][j+1]);
  fprintf(Fout,"\\}) $ \\\\ \\hline\n");
  for(j=0;j<g;j++)
  {
    fprintf(Fout,"$H_{%d}: ",j);
      for(i=0;i<v-1;i++)
      {
        fscanf(Fin," %d ",&i0);
        a0=i0%na;
        b0=(i0-a0)/na;
        fprintf(Fout,"(%d,%d),",a0,b0);
        if( i == 18 || i == 36)
          fprintf(Fout, "$ \\\\ \\hspace{.16in}$ ");
      }
      fscanf(Fin," %d ",&i0);
      a0=i0%na;
      b0=(i0-a0)/na;
      fprintf(Fout, "(%d,%d)$ \\\\ \n",a0,b0);
  }
   fprintf(Fout, "\\hline \n");
 }
 free(K);
 free(H);
 close(Fout);
 close(Fin);
 close(Fgens);
}

/*-----------------------Convert3.c-----------------------*/
#include<stdlib.h>
#include<stdio.h>

FILE *Fin, *Fgens, *Fout;
int * H, **K;
int n,na,nb,nc,m,s,v;
int S[10][10][10];

main( int ac, char * av[])
{
  int i,j,g,k,counter;
```

```
 int i0,j0,k0;
 int i1,a1,b1;
 if(ac!=4)
 {
  fprintf(stderr,"Usage: %s HCYC fgens fout\n",av[0]);
  exit(1);
 }
if( !(Fin=fopen(av[1],"r")) )
{
  fprintf(stderr,"%s cannot open %s\n", av[0],av[1]);
  exit(1);
}
if( !(Fgens=fopen(av[2],"r")) )
{
  fprintf(stderr,"%s cannot open %s\n", av[0],av[2]);
  exit(1);
}
if( !(Fout=fopen(av[3],"w")) )
{
  fprintf(stderr,"%s cannot open %s\n", av[0],av[3]);
  exit(1);
}
fscanf(Fgens, "%d %d %d %d %d\n",&n,&na,&nb,&nc,&g);
v=na*nb*nc;

counter = 0;
 for(i0=0;i0<na;i0++)
   for(j0=0;j0<nb;j0++)
          for(k0=0;k0<nc;k0++)
          {
                S[i0][j0][k0] = counter;
                counter++;
          }

H = (int *)calloc(v,sizeof(int));
K = (int **)calloc(n,sizeof(int*));

for(i=0;i<n;i++)
{
  K[i] = (int*)calloc(3*g,sizeof(int));
  for(j=0;j<3*g;j++)
    fscanf(Fgens, "%d ", &K[i][j]);
}
for(k=0;k<n;k++)
{
 fprintf(Fout,
 "$\\cay(\\zed_{%d}\\times\\zed_{%d}\\times\\zed_{%d},"
 "\\{(%d,%d,%d),(%d,%d,%d),(%d,%d,%d)\\})$\\\\\\hline\n"
 ,na,nb,nc,
 K[k][0],K[k][1],K[k][2],
 K[k][3],K[k][4],K[k][5],
 K[k][6],K[k][7],K[k][8]);
 for(j=0;j<g;j++)
 {
```

```
   fprintf(Fout, "$H_{%d}: ", j);
   for(i=0;i<v;i++)
   {
    fscanf(Fin," %d ", &i1 );
        for(i0=0;i0<na;i0++)
         for(j0=0;j0<nb;j0++)
          for(k0=0;k0<nc;k0++)
           if( S[i0][j0][k0] == i1 )
           {
                if( i != v-1)
                {
                 fprintf(Fout, "(%d,%d,%d),",i0,j0,k0);
                 break;
                }else
                 fprintf(Fout, "(%d,%d,%d)$\\\\ \n",i0,j0,k0);
           }
           if( i == 14 || i == 28)
                fprintf(Fout, "\n$\\\\\\hspace{.16in}$ ");
   }
  }
   fprintf(Fout, "\\hline \n");
 }
 free(K);
 free(H);
 close(Fout);
 close(Fin);
 close(Fgens);
}
```

2.3 Shell Scripts

The following shell scripts were used to find random Hamilton cycle decompositions for the graphs obtained using MAGMA in Section 2.1. They were written with Donald L. Kreher.

```
#! /bin/csh
if ( $#argv != 2 ) then
  echo "usage:  RUN GeneratorFile NumberOfGroups "
endif
if ( $2 == 1 ) then
./GetGraph $1 > G1
else if ( $2 == 2 ) then
./GetGraph2 $1 > G1
else if ( $2 == 3 ) then
./GetGraph3 $1 > G1
endif
set h3=0
while ( $h3 != 1 )
set h2=0
while ( $h2 != 1 )
./RHC G1 > H1;
./DelHCyc G1 H1 > G2
set h1=`wc -l H1`
```

```
set h1=`echo $h1 | sed -e 's/ .*//'`
if ( $h1 != 1 ) then
echo  "Not connected" >> CycleList
exit
endif
./RHC G2 > H2;
./DelHCyc G2 H2 > G3
set h2=`wc -l H2`
set h2=`echo $h2 | sed -e 's/ .*//'`
end
./RHC G3 > H3 ;
set h3=`wc -l H3`
set h3=`echo $h3 | sed -e 's/ .*//'`
end
cat H1 >> $1
cat H2 >> $1
cat H3 >> $1
cat H1 >> CycleList
cat H2 >> CycleList
cat H3 >> CycleList

echo " " >> CycleList
```

```
#! /bin/csh
if ( $#argv != 3 ) then
  echo "usage:  RUN CycleList SetofGenerators NumberOfGroups"
endif
if ( $3 == 1 ) then
./Convert1 $1 $2 LaTeXOut
else if ( $3 == 2 ) then
./Convert2 $1 $2 LaTeXOut
else if ( $3 == 3 ) then
./Convert3 $1 $2 LaTeXOut
endif
```

2.4 Mathematica Code

The following is Mathematica code written (with Matthew Miller and Raymond Molzon) to produce the graphics in Figure 2.1.

```
b:=2π(n−1)/n;
baseT:=RevolutionPlot3D[{2 + Cos[t], Sin[t]}, {t, 0, 2π}, {θ, 0, 2π},
ColorFunction → "ArmyColors",
Mesh → None];
Edges:=RevolutionPlot3D[{2 + Cos[t], Sin[t]}, {t, 0, 2π}, {θ, 0, b}, PlotStyle → None,
Mesh → {Range[0, 2π, 2π/m], Range[0, b, b/(n − 1)]},
MeshStyle → {Directive[Blue, Thick], Directive[Black, Thick]}];
kludge:=RevolutionPlot3D[{{3, 0}, {3.01, 0}}, {t, 0, 2π}, {θ, 0, b}, PlotStyle → None,
MeshStyle → Directive[Blue, Thick]];
Pt[a_]:={2 + Cos[a], 0, Sin[a]};
```

```
bpts:=Table[Pt[i*2π/m], {i, 0, m}];
rotr[X_, i_]:=RotationMatrix[i*b/(n-1), {0, 0, 1}].X;
pts:=Table[rotr[bpts[[j]], i], {i, 0, n-1}, {j, 1, m}];
Sphr[X_]:=Sphere[X, 0.075];
Vertices:=Graphics3D[{Black, Map[Sphr, Flatten[pts, 1]]}];
θ[φ_, i_]:=(nr)/m(φ-2π) + (2π(i-1+r))/m
x[φ_, i_]:=Cos[φ](2 + Cos[θ[φ, i]])
y[φ_, i_]:=Sin[φ](2 + Cos[θ[φ, i]])
z[φ_, i_]:=- Sin[θ[φ, i]]
JumpEdges:=
Table[ParametricPlot3D[{x[p, j], y[p, j], z[p, j]}, {p, b, 2π},
PlotStyle → Directive[Blue, Thick]],
{j, 1, m}]

n = 10;

(*Numberofcolumnsinred2 - factor*)
m = 8; (*Numberofrowsinblue2 - factor*)
r = 2; (* The Jump Number *)
GCD[r, m]; (*Numberofhorizontalcycles.*)
torus = Show[baseT, Edges, JumpEdges, kludge, Vertices, Axes → None, Boxed → False,
PlotRange → {-1.05, 1.05}]
```

References

[1] Alspach, B. (1979). Hamiltonian cycles in vertex-transitive graphs of order $2p$. In *Proceedings of the Tenth Southeastern Conference on Combinatorics, Graph Theory and Computing (Florida Atlantic Univ., Boca Raton, Fla., 1979)*. Congress. Numer., XXIII–XX. Utilitas Math., Winnipeg, Man., 131–139. MR561039

[2] Alspach, B. (1984). Research problem 59. *Discrete Math. 50*, 115.

[3] Alspach, B. (1989). Lifting Hamilton cycles of quotient graphs. *Discrete Math.* **78**, 1-2, 25–36. http://dx.doi.org/10.1016/0012-365X(89)90157-X. MR1020643 (91c:05091)

[4] Alspach, B., Bermond, J.-C., and Sotteau, D. (1990). Decomposition into cycles. I. Hamilton decompositions. In *Cycles and rays (Montreal, PQ, 1987)*. NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., Vol. **301**. Kluwer Acad. Publ., Dordrecht, 9–18. MR1096980

[5] Alspach, B., Dyer, D., and Kreher, D. L. (2006). On isomorphic factorizations of circulant graphs. *J. Combin. Des.* **14**, 5, 406–414. MR2245555

[6] Alspach, B. and Zhang, C. Q. (1989). Hamilton cycles in cubic Cayley graphs on dihedral groups. *Ars Combin. 28*, 101–108. MR1039136 (91b:05124)

[7] Aubert, J. and Schneider, B. (1982). Décomposition de la somme cartésienne d'un cycle et de l'union de deux cycles hamiltoniens en cycles hamiltoniens. *Discrete Math.* **38**, 1, 7–16. MR676514

[8] Beineke, L. W. and Wilson, R. J., Eds. (2004). *Topics in algebraic graph theory*. Encyclopedia of Mathematics and its Applications, Vol. **102**. Cambridge University Press, Cambridge. MR2125091

[9] Bermond, J.-C., Favaron, O., and Mahéo, M. (1989). Hamiltonian decomposition of Cayley graphs of degree 4. *J. Combin. Theory Ser. B* **46**, 2, 142–153. MR992988

[10] Bondy, J. A. and Chvátal, V. (1976). A method in graph theory. *Discrete Math.* **15**, 2, 111–135. MR414429

[11] Chen, C. C. and Quimpo, N. F. (1981). On strongly Hamiltonian abelian group graphs. In *Combinatorial mathematics, VIII (Geelong, 1980)*. Lecture Notes in Math., Vol. **884**. Springer, Berlin, 23–34. MR641233

[12] Chen, Y. Q. (1998). On Hamiltonicity of vertex-transitive graphs and digraphs of order p^4. *J. Combin. Theory Ser. B* **72**, 1, 110–121. http://dx.doi.org/10.1006/jctb.1997.1796. MR1604701

[13] Chvátal, V. and Erdős, P. (1972). A note on Hamiltonian circuits. *Discrete Math. 2*, 111–113. MR0297600

[14] Curran, S. J. and Gallian, J. A. (1996). Hamiltonian cycles and paths in Cayley graphs and digraphs—a survey. *Discrete Math.* **156**, 1-3, 1–18. MR1405010

[15] Dean, M. (2006). On Hamilton cycle decomposition of 6-regular circulant graphs. *Graphs Combin.* **22**, 3, 331–340. MR2264855

[16] Dean, M. (2007). Hamilton cycle decomposition of 6-regular circulants of odd order. *J. Combin. Des.* **15**, 2, 91–97. MR2291522

[17] Diestel, R. (2005). *Graph theory*, Third ed. Graduate Texts in Mathematics, Vol. **173**. Springer-Verlag, Berlin. MR2159259

[18] Dirac, G. A. (1952). Some theorems on abstract graphs. *Proc. London Math. Soc. (3) 2*, 69–81. MR0047308

[19] Dobson, E., Gavlas, H., Morris, J., and Witte, D. (1998). Automorphism groups with cyclic commutator subgroup and Hamilton cycles. *Discrete Math.* **189**, 1-3, 69–78. http://dx.doi.org/10.1016/S0012-365X(98)00003-X. MR1637709

[20] Durnberger, E. (1985). Every connected Cayley graph of a group with prime order commutator group has a Hamilton cycle. In *Cycles in graphs (Burnaby, B.C., 1982)*. North-Holland Math. Stud., Vol. **115**. North-Holland, Amsterdam, 75–80. MR821506

[21] Fan, C., Lick, D. R., and Liu, J. (1996). Pseudo-Cartesian product and Hamiltonian decompositions of Cayley graphs on abelian groups. *Discrete Math.* **158**, 1-3, 49–62. MR1411109

[22] Fleischner, H. (1974a). On spanning subgraphs of a connected bridgeless graph and their application to *DT*-graphs. *J. Combinatorial Theory Ser. B 16*, 17–28. MR0332572

[23] Fleischner, H. (1974b). The square of every two-connected graph is Hamiltonian. *J. Combinatorial Theory Ser. B 16*, 29–34. MR0332573

[24] Gould, R. J. (2003). Advances on the Hamiltonian problem—a survey. *Graphs Combin.* **19**, 1, 7–52. MR1974368

[25] Keating, K. and Witte, D. (1985). On Hamilton cycles in Cayley graphs in groups with cyclic commutator subgroup. In *Cycles in graphs (Burnaby, B.C., 1982)*. North-Holland Math. Stud., Vol. **115**. North-Holland, Amsterdam, 89–102. MR821508

[26] Kreher, D. L. and Westlund, E. E. (2010). n-Isofactorizations of 8-regular circulant graphs. *J. Combin. Math. Combin. Comput. 67*, 197–209.

[27] Kutnar, K. and Marušič, D. (2008). Hamiltonicity of vertex-transitive graphs of order $4p$. *European J. Combin.* **29**, 2, 423–438. http://dx.doi.org/10.1016/j.ejc.2007.02.002. MR2388379

[28] Kutnar, K. and Marušič, D. (2009). Hamilton cycles and paths in vertex transitive graphs—current directions. *Discrete Math.* **309**, 17, 5491–5500. http://dx.doi.org/10.1016/j.disc.2009.02.017. MR2548567

[29] Kutnar, K. and Šparl, P. (2009). Hamilton paths and cycles in vertex-transitive graphs of order $6p$. *Discrete Math.* **309**, 17, 5444–5460. http://dx.doi.org/10.1016/j.disc.2008.12.005. MR2548563

[30] Li, H., Wang, J., and Sun, L. (2000). Hamiltonian decomposition of Cayley graphs of orders p^2 and pq. *Acta Math. Appl. Sinica (English Ser.)* **16**, 1, 78–86. MR1757325

[31] Li, Y. Q. and Yao, B. H. (1989). A note on the Hamilton decomposition of Cayley graphs on noncommutative groups. *Chinese Quart. J. Math.* **4**, 1, 82–86. MR1014109 (90g:05143)

[32] Liu, J. (1994). Hamiltonian decompositions of Cayley graphs on abelian groups. *Discrete Math.* **131**, 1-3, 163–171. MR1287730

[33] Liu, J. (1996). Hamiltonian decompositions of Cayley graphs on abelian groups of odd order. *J. Combin. Theory Ser. B* **66**, 1, 75–86. MR1368517

[34] Liu, J. (2003). Hamiltonian decompositions of Cayley graphs on abelian groups of even order. *J. Combin. Theory Ser. B* **88**, 2, 305–321. MR1983361

[35] Locke, S. C. and Witte, D. (1999). On non-Hamiltonian circulant digraphs of outdegree three. *J. Graph Theory* **30**, 4, 319–331. http://dx.doi.org/10.1002/(SICI)1097-0118(199904)30:4*319::AID-JGT6*3.3.CO;2-T. MR1669452

[36] Lovász, L. (1970). The factorization of graphs. In *Combinatorial Structures and their Applications (Proc. Calgary Internat. Conf., Calgary, Alta., 1969)*. Gordon and Breach, New York, 243–246. MR0268071

[37] Lovász, L. (2007). *Combinatorial problems and exercises*, second ed. AMS Chelsea Publishing, Providence, RI. MR2321240

[38] Marušič, D. (1983). Hamiltonian circuits in Cayley graphs. *Discrete Math.* **46**, 1, 49–54. MR708161

[39] Marušič, D. (1984). Some problems in vertex symmetric graphs. In *Finite and infinite sets, Vol. I, II (Eger, 1981)*. Colloq. Math. Soc. János Bolyai, Vol. **37**. North-Holland, Amsterdam, 581–589. MR818261

[40] Marušič, D. (1985). Vertex transitive graphs and digraphs of order p^k. In *Cycles in graphs (Burnaby, B.C., 1982)*. North-Holland Math. Stud., Vol. **115**. North-Holland, Amsterdam, 115–128. http://dx.doi.org/10.1016/S0304-0208(08)73001-9. MR821510

[41] Marušič, D. (1987). Hamiltonian cycles in vertex symmetric graphs of order $2p^2$. *Discrete Math.* **66**, 1-2, 169–174. http://dx.doi.org/10.1016/0012-365X(87)90129-4. MR900940 (88f:05074)

[42] Marušič, D. and Parsons, T. D. (1982). Hamiltonian paths in vertex-symmetric graphs of order $5p$. *Discrete Math.* **42**, 2-3, 227–242. http://dx.doi.org/10.1016/0012-365X(82)90220-5. MR677056

[43] Marušič, D. and Parsons, T. D. (1983). Hamiltonian paths in vertex-symmetric graphs of order $4p$. *Discrete Math.* **43**, 1, 91–96. http://dx.doi.org/10.1016/0012-365X(83)90024-9. MR680307

[44] McKay, B. D. and Praeger, C. E. (1994). Vertex-transitive graphs which are not Cayley graphs. I. *J. Austral. Math. Soc. Ser. A* **56**, 1, 53–63. MR1250993

[45] Nash-Williams, C. S. J. A. (1971). Hamiltonian arcs and circuits. In *Recent Trends in Graph Theory (Proc. Conf., New York, 1970)*. Lecture Notes in Mathematics, Vol. 186. Springer, Berlin, 197–210. MR0277417

[46] Ore, O. (1960). Note on Hamilton circuits. *Amer. Math. Monthly 67*, 55. MR0118683

[47] Pak, I. and Radoičić, R. (2009). Hamiltonian paths in cayley graphs. *Discrete Math.* **309**, 17, 5501–5508.

[48] Sabidussi, G. (1958). On a class of fixed-point-free graphs. *Proc. Amer. Math. Soc. 9*, 800–804. MR0097068

[49] Sabidussi, G. (1964). Vertex-transitive graphs. *Monatsh. Math. 68*, 426–438. MR0175815

[50] Stong, R. (1991). Hamilton decompositions of Cartesian products of graphs. *Discrete Math.* **90**, 2, 169–190. MR1115502

[51] Turner, J. (1967). Point-symmetric graphs with a prime number of points. *J. Combinatorial Theory 3*, 136–145. MR0211908

[52] West, D. B. (1996). *Introduction to graph theory.* Prentice Hall Inc., Upper Saddle River, NJ. MR1367739

[53] Westlund, E. E., Liu, J., and Kreher, D. L. (2009). 6-regular cayley graphs on abelian groups of odd order are hamiltonian decomposable. *Discrete Math.* **309**, 16, 5106–5110.

[54] Witte, D. (1986). Cayley digraphs of prime-power order are Hamiltonian. *J. Combin. Theory Ser. B* **40**, 1, 107–112. MR830597

[55] Witte, D. and Gallian, J. A. (1984). A survey: Hamiltonian cycles in Cayley graphs. *Discrete Math.* **51**, 3, 293–304. http://dx.doi.org/10.1016/0012-365X(84)90010-4. MR762322

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